Parametrised categories and categories by proxy

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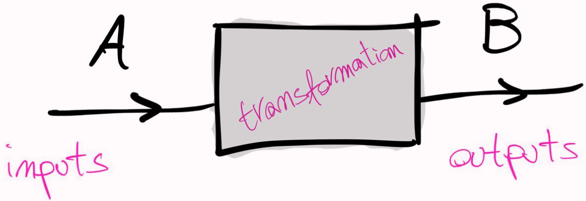
Motivation: Para

It is now established that symmetric monoidal categories $\mathcal{C}$ model resource/process theories:



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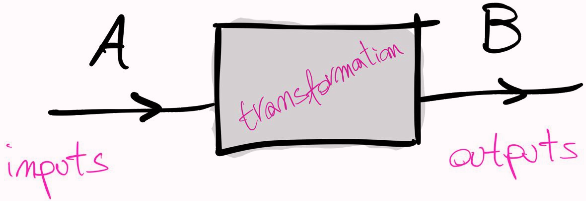

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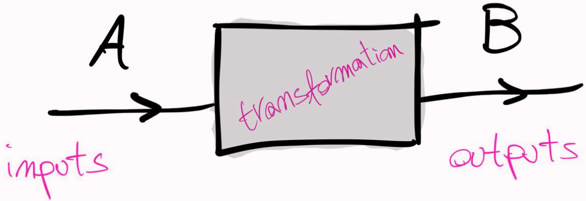

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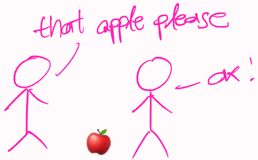

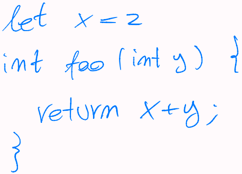

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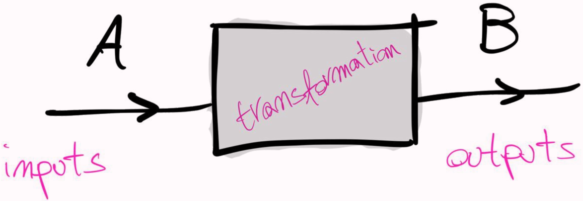

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1. context (e.g. natural language, programs [POM14])
2. private state (e.g. automata [KSW97])

```
int foo (int y) {
  static int x = y*y;
  return x+y;
}
```

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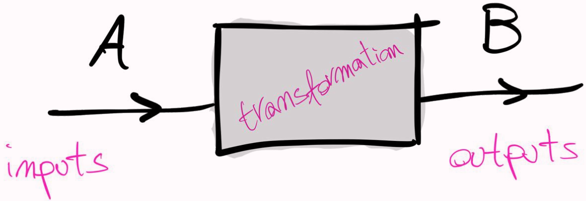

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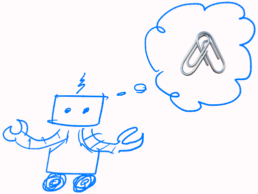

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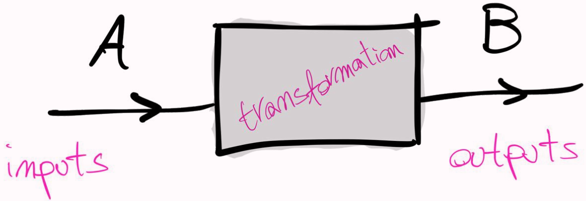

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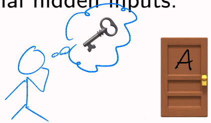

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3. policies/controls (e.g. games [Cap+21b], cybernetic systems [Cap+21a; FST19; Cru+21])
4. guards/keys (e.g. secure processes [BK21], chemical reactions)

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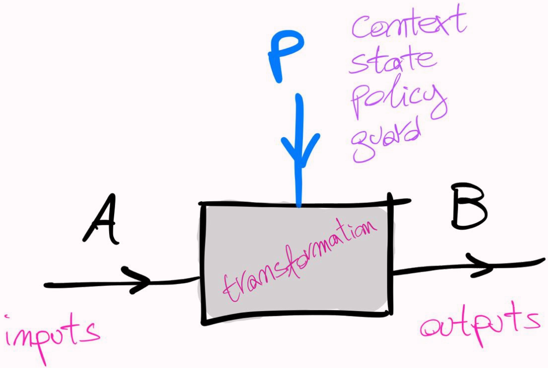
It becomes natural, therefore, to add a third (possibly even a fourth [KSW97]) port to processes:



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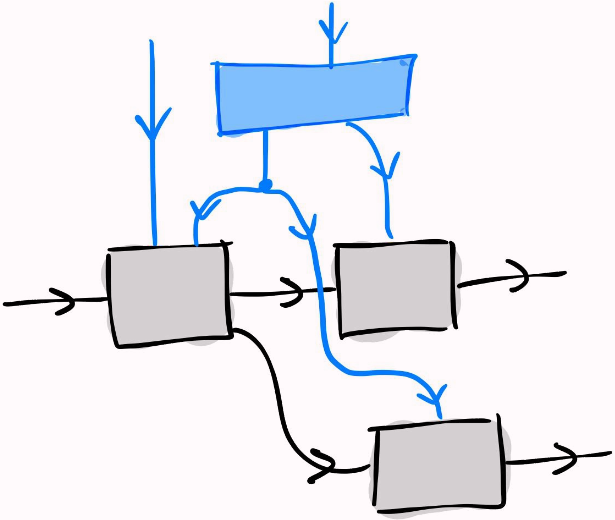
These additional ports live in a orthogonal direction and are part of their own resource theory $\mathcal{M}$:



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1. $\mathcal{C}(A, B)$ is too 'big' (e.g. ∞-dimensional, as for Smooth(A, B))
2. a process in $\mathcal{C}(A, B)$ is non-deterministically chosen (e.g. stochastic processes, [Smi20])

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already caught a new example in yesterday talks!

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3. Where could we go from here?

Fundamental idea

Suppose we work in a monoidal $\mathcal{V}$-cat $\mathcal{C}$.

Parametrised functions are, essentially, 'families of maps' $A \to B$ picked out by a parameter $p \in P$:

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Very different! Compare:

a smooth family of maps $f : \mathrm{Smooth}(P \times A, B)$

vs.

a family of smooth maps $f : \mathrm{Set}(P, \mathrm{Smooth}(A, B))$

Para

Oplax $\mathcal{M}$-modules

Let $(\mathcal{M}, \odot, J)$ be a (small enough) monoidal category.

Definition

A **strict left oplax $\mathcal{M}$-module** is a 2-category $\mathcal{C}$ equipped with

an *action* $- \bullet - : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$, a *counitor* $\varepsilon : J \bullet X \to X$,

a *comultiplicator* $\delta : (M \odot N) \bullet X \to M \bullet (N \bullet X)$

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In most cases, $\mathcal{C}$ is mon itself and $- \bullet - := F(-) \otimes -$ from a functor $F : \mathcal{M} \to \mathcal{C}$.
(e.g. F is the left part of a LNL adjunction).

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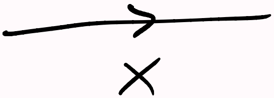
$\mathsf{Para}_\bullet(\mathcal{C})$ is a bicategory defined by the following data:

a. objects are the same of $\mathcal{C}$,



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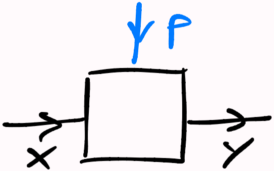
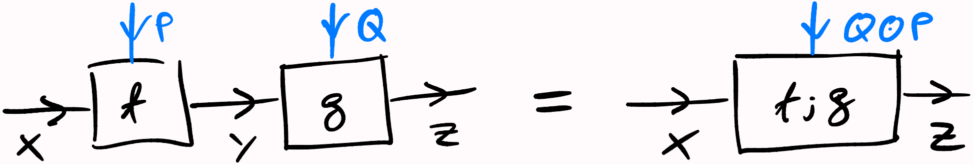


with identities provided by ε:

$$1_X : X \to X := (J, \varepsilon_X : J \bullet X \to X)$$

and composition mediated by δ:

$$(P, f : P \bullet X \to Y) \,\mathbin{⨾}\, (Q, g : Q \bullet Y \to Z) \;:=\; (Q \odot P, (Q \odot P) \bullet X \xrightarrow{\delta} Q \bullet (P \bullet X) \xrightarrow{Q \bullet f} Q \bullet Y \xrightarrow{g} Z)$$



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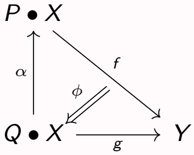
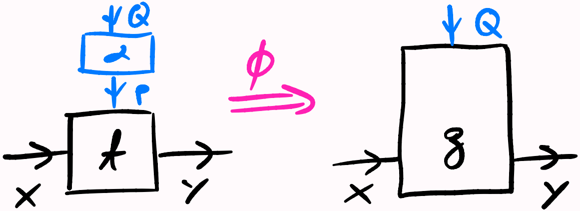
c. 2-morphisms are given by pairs $(\alpha, \phi) : (P, f) \Rightarrow (Q, g) : X \to Y$ where $\alpha : Q \to P$ in $\mathcal{M}$ and $\phi : (\alpha \bullet X) \mathbin{⨟} f \Rightarrow g$ is a 2-morphism of $\mathcal{C}$:





Notice: if $\mathcal{C}$ is a discrete 2-category, the second part of a 2-morphism is always trivial.

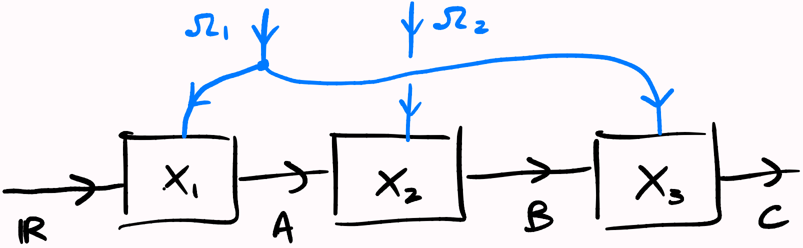
Examples

Example

Let $\mathcal{C} := \mathsf{Msbl}$, $\mathcal{M} := \mathsf{Prob}$ acting by cartesian product:

$$(\Omega, \mathcal{F}, \mathbb{P}) \bullet (X, \Sigma_X) := (\Omega \times X, \mathcal{F} \otimes \Sigma_X)$$

Then $\mathbf{Para}_\bullet(\mathsf{Msbl})$ is a category of stochastic processes with explicit 'stochastic bases'. 2-cells can be used to manipulate randomness sources (e.g. use Δ_Ω to correlate two processes).



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Example

Let $\mathcal{C} := \mathsf{Met}$, $\mathcal{M} := (\mathbb{R}, \leq, 1, \cdot)$ acting by dilation:

$$r \bullet (X, d) = (X, r \cdot d)$$

$\forall x, y \in Y \quad d'(x,y) \leq L d(x,y)$

Then $\mathsf{Para}_\bullet(\mathsf{Met})$ is a ‘proof-relevant’ version of Lip, i.e. maps $(L, f) : (X, d) \to (Y, d')$ are Lipschitz maps with an explicit choice of constant L. 2-cells are refinements of constants.

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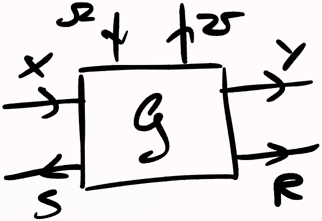
Parametrised optics [Cap+21a] are our current proposal for a foundation of categorical cybernetics.



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Example

Also from yesterday's talk by JS Pacaud-Lemay [HL20], the $\mathcal{S}[-]$ construction on a monoidal category $\mathcal{C}$ is $\mathsf{Para}_\otimes(\mathcal{C})$, where $\mathcal{M}$ = subunits of $\mathcal{C}$, acting through $\mathcal{C}$'s own $\otimes$.

Warning! In $\mathcal{S}[-]$ 2-cells are quotiented away...

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In categorical cybernetics, 2-morphisms ('**reparametrisations**') are crucial, see [Cap+21a] and [Cap21].

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In categorical cybernetics, 2-morphisms (‘**reparametrisations**’) are crucial, see [Cap+21a] and [Cap21].

Warning! Spivak [Spi21] uses 2-cells but of a different kind.

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To do this, we need $\mathcal{C}$ to be itself monoidal $(I, \otimes, \ldots)$ and $\bullet$ to be compatible with this structure:

Definition

A **costrength** for an oplax $\mathcal{M}$-module is a natural transformation

$$\kappa : M \bullet (X \otimes Y) \longrightarrow X \otimes (M \bullet Y)$$

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Proposition

If $\mathcal{C}$ is symmetric monoidal and a costrong oplax $\mathcal{M}$-module,
then $\mathrm{Para}_\bullet(\mathcal{C})$ *is itself symmetric monoidal.*

Functoriality

Strict left oplax $\mathcal{M}$-modules $(\mathcal{C}, \bullet), (\mathcal{D}, \circ), \ldots$ gather in a tricategory $\mathcal{M}\text{-}\mathrm{ox}\mathbb{M}\mathrm{od}_{\mathrm{lx}}$:

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b. 2-morphisms are **linear natural transformations**, i.e. natural transformations $\beta : F \Rightarrow G$ such that

$$\begin{array}{ccc} M \circ F(X) & \xrightarrow{M \circ \beta_X} & M \circ G(X) \\ \downarrow{\scriptstyle \ell_F} & & \downarrow{\scriptstyle \ell_G} \\ F(M \bullet X) & \xrightarrow{\beta_{M \bullet X}} & G(M \bullet X) \end{array}$$

Functoriality

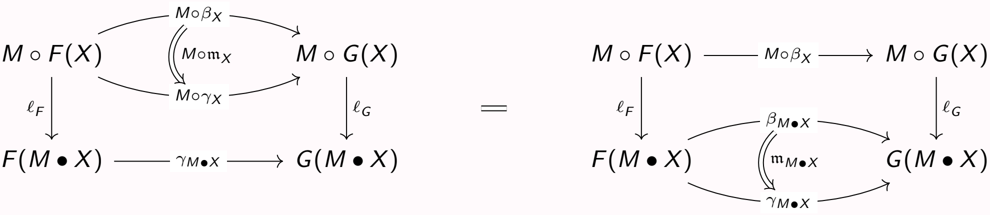
Strict left oplax $\mathcal{M}$-modules $(\mathcal{C}, \bullet), (\mathcal{D}, \circ), \ldots$ gather in a tricategory $\mathcal{M}\text{-oxMod}_{\mathrm{lx}}$:

a. 1-morphisms are **lax linear functors**, i.e. functors $F : \mathcal{C} \to \mathcal{D}$ equipped with a lineator $\ell : M \circ F(X) \longrightarrow F(M \bullet X)$ satisfying suitable coherences.

b. 2-morphisms are **linear natural transformations**, i.e. natural transformations $\beta : F \Rightarrow G$ such that

$$\begin{array}{ccc} M \circ F(X) & \xrightarrow{M \circ \beta_X} & M \circ G(X) \\ {\scriptstyle \ell_F}\downarrow & & \downarrow{\scriptstyle \ell_G} \\ F(M \bullet X) & \xrightarrow{\beta_{M \bullet X}} & G(M \bullet X) \end{array}$$

c. 3-morphisms are **linear modifications** $\mathfrak{m} : \beta \Rrightarrow \gamma$ commuting with ℓ in a suitable sense:



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Proposition

If $\mathcal{M}$ *is symmetric,* $\text{Para} : \mathcal{M}\text{-psMod}_{\text{lx}} \to \mathcal{M}\text{-psMod}_{\text{lx}}$ *is a 2-monad:*

its unit $\eta : 1 \Rightarrow \text{Para}$ *trivially parametrise morphisms:*

$$f : X \to Y \quad \longmapsto \quad (J, \varepsilon \mathbin{⨟} f)$$

its multiplication $\mu : \text{Para} \mathbin{⨟} \text{Para} \Rightarrow \text{Para}$ *multiplies parameters together:*

$$(P, (Q, f)) : X \to Y \quad \longmapsto \quad (Q \odot P, \delta \mathbin{⨟} f)$$

This remains true if we co/restrict to symmetric costrong oplax $\mathcal{M}$-modules.

Universal property

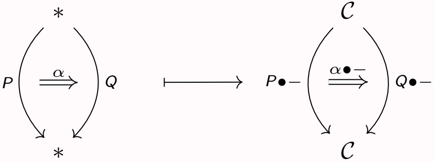
Suppose $\mathcal{C}$ is a pseudo-$\mathcal{M}$-module. $\mathrm{Para}_\bullet(\mathcal{C})$ satisfies a 3-dimensional universal property:

Proposition

$\mathrm{Para}_\bullet(\mathcal{C})$ *is the 2-Grothendieck construction of the delooping of* $\bullet$*:*

$$\mathbb{B}(\bullet) : \mathbb{B}\mathcal{M} \longrightarrow 2\mathbb{C}\mathrm{at}$$

defined as follows:



Proof.

Compare **Para** definition with the 2-Grothendieck construction here [Bak]. □

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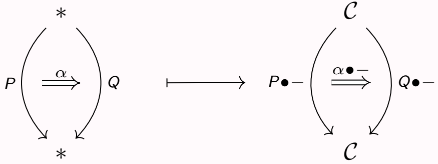

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Is it useful to make this the definition, possibly allowing arbitrary bicats in place of $\mathbb{B}\mathcal{M}$?

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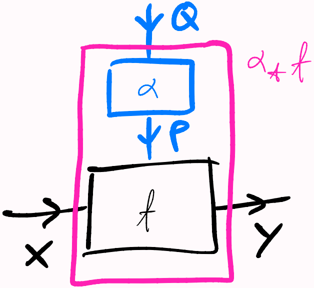
Moreover – and this is important for applications:

Proposition (Reparametrisation)

$\mathsf{Para}_\bullet(\mathcal{C})$ *is locally opfibred over* $\mathcal{M}^{\mathsf{op}}$*:*

Proof.

$$\begin{array}{ccc} (P, f) & \xrightarrow{(\alpha,1)} & (Q, \alpha_* f) \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ P & \xleftarrow{\alpha} & Q \end{array}$$



where $\alpha_* f = (\alpha \bullet X) \fatsemi f$. □

Proxy

Definition

We work in an explicitly $\mathcal{V}$-enriched context here. For now, $(\mathcal{V}, I, \boxtimes)$ is only (strictly) monoidal.

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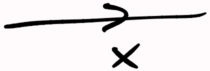

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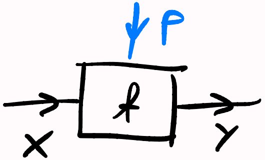
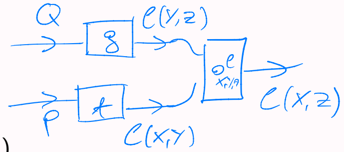


$$\mathrm{Proxy}(\mathcal{C})(X, Y) = \sum_{P:\mathcal{V}} \mathcal{V}(P, \mathcal{C}(X, Y))$$

with identity given by the enrichment structure:

$$(I, 1_X : I \to \mathcal{C}(X, Y))$$



and composition likewise:

$$X \xrightarrow{(P,f)} Y \xrightarrow{(Q,g)} Z := X \xrightarrow{(Q \boxtimes P, (g \boxtimes f) \fatsemi \circ^{\mathcal{C}}_{X,Y,Z})} Z$$

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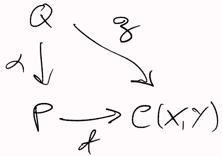

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Notice: if $\mathcal{V}$ is monoidal closed and cocomplete, then we can replace $\mathcal{V}(-,-)$ with $[-,-]_{\mathcal{V}}$ and (truncated) **Proxy** becomes an endofunctor (actually, a 2-monad) $\mathcal{V}$-$\mathbb{C}$at $\to$ $\mathcal{V}$-$\mathbb{C}$at.

Example

Let $\mathcal{C}$ be a Set-category. Consider covariant powerset $\mathscr{P} : \mathsf{Set} \to \mathsf{Set}$. It's a monoidal monad.

Change base along $\mathsf{Set} \to \mathrm{Kl}(\mathscr{P})$ and run **Proxy**, we obtain a category of non-deterministic choices of morphisms $\mathsf{Proxy}(\mathscr{P}_*\mathcal{C})$.

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Let $\mathcal{C}$ be a Meas^σ-category. Consider Giry monad $\mathscr{D} : \mathsf{Meas}^\sigma \to \mathsf{Meas}^\sigma$. It's a monoidal monad. Change base along $\mathsf{Meas}^\sigma \to \mathrm{Kl}(\mathscr{D})$ and run **Proxy**, we obtain a category of probabilistic choices of morphisms $\mathsf{Proxy}(\mathscr{D}_* \mathcal{C})$.

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General pattern: $\mathscr{T}$ monoidal monad on $\mathcal{V}$, change base along $\mathcal{V} \to \mathrm{Kl}(\mathscr{T})$ and run **Proxy** to get a cat of $\mathscr{T}$-effectful choices of morphisms.

A one-line construction

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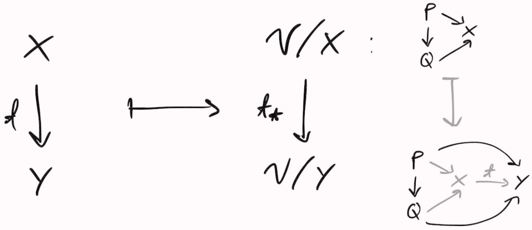
Proxy($\mathcal{C}$) *is the change of base of* $\mathcal{C}$ *along the covariant slice functor* $\mathcal{V}/-$*:*

$$\mathcal{V}/- \; : \; \mathcal{V} \longrightarrow \mathbb{C}\mathrm{at}$$



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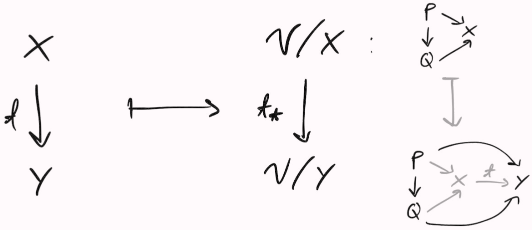

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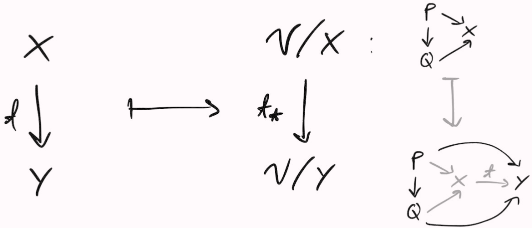

Hence one can consider Proxy($\mathcal{C}$) the underlying 2-category of an enriched category.

In other words, we recover a 2-category by considering **generalised elements** of $\mathcal{V}/\mathcal{C}(X, Y)$ instead of only the global points $I \to \mathcal{C}(X, Y)$.

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Proof.

1. See [GP97]: closed $\mathcal{V}$-modules are equivalent to tensored $\mathcal{V}$-cats. The pra $[-, -]$ provides the enrichment.
2. The equivalence is witnessed by an identity-on-object functor which acts on morphisms by transposition along the adjunction $- \bullet X \vdash [X, -]$:

$$\mathrm{Para}_\bullet(\mathcal{C})(X, Y) = \sum_{P:\mathcal{M}} \mathcal{C}(P \bullet X, Y) \cong \sum_{P:\mathcal{M}} \mathcal{M}(P, \underbrace{[X, Y]}_{=:\mathcal{C}(X,Y)}) = \mathrm{Proxy}(\mathcal{C})(X, Y)$$

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Both have interesting higher-dimensional structure to deal with dynamics in the parameters.

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3. Can we relate **Para** to the graded coKleisli construction?
   In fact oplax $\mathcal{M}$-modules are the same as $\mathcal{M}$-graded comonads, and the coKleisli object is an oplax colimit itself. Again, hard to reconcile with 2-cells (also [Fuj19] has a very different construction!)

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5. Iterated versions: what happens when reparametrisations are themselves parametrised?
   Modelling 'nested systems', see [Smi21b] and [Cap21].

Thanks for your attention!

Questions?

References I

- I. Bakovic, "Grothendieck construction for bicategories", *Preprint available at http://www.irb.hr/users/ibakovic/sgc.pdf*,
- A. Broadbent and M. Karvonen, "Categorical composable cryptography", *arXiv preprint arXiv:2105.05949*, 2021.
- M. Capucci. (2021), Open cybernetic systems ii: Parametrised optics and agency, Blog article, [Online]. Available: https://matteocapucci.wordpress.com/2021/06/21/open-cybernetic-systems-ii-parametrised-optics-and-agency/.
- M. Capucci, B. Gavranović, J. Hedges, and E. F. Rischel, "Towards foundations of categorical cybernetics", *arXiv preprint arXiv:2105.06332*, 2021.
- M. Capucci, N. Ghani, J. Ledent, and F. N. Forsberg, "Translating extensive form games to open games with agency", *arXiv preprint arXiv:2105.06763*, 2021.
- G. S. Cruttwell, B. Gavranović, N. Ghani, P. Wilson, and F. Zanasi, "Categorical foundations of gradient-based learning", *arXiv preprint arXiv:2103.01931*, 2021.

References II

- D. A. Dalrymple, *Dioptics: A common generalization of gradient-based learners and open games*, 2019. [Online]. Available: https://www.cl.cam.ac.uk/events/syco/strings3-syco5/slides/dalrymple.pdf.
- B. Fong, D. Spivak, and R. Tuyéras, "Backprop as functor: A compositional perspective on supervised learning", in *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, IEEE, 2019, pp. 1–13.
- S. Fujii, "A 2-categorical study of graded and indexed monads", *arXiv preprint arXiv:1904.08083*, 2019.
- R. Gordon and A. J. Power, "Enrichment through variation", *Journal of Pure and Applied Algebra*, vol. 120, no. 2, pp. 167–185, 1997.
- J. W. Gray, "Closed categories, lax limits and homotopy limits", *Journal of Pure and Applied Algebra*, vol. 19, pp. 127–158, 1980.
- C. Heunen and J. Lemay, "Tensor-restriction categories", *arXiv preprint arXiv:2009.12432*, 2020.

References III

P. Katis, N. Sabadini, and R. F. Walters, "Bicategories of processes", *Journal of Pure and Applied Algebra*, vol. 115, no. 2, pp. 141–178, 1997.

T. Petricek, D. Orchard, and A. Mycroft, "Coeffects: A calculus of context-dependent computation", in *Proceedings of International Conference on Functional Programming*, ser. ICFP 2014, Gothenburg, Sweden, 2014.

T. S. C. Smithe, "Bayesian updates compose optically", *arXiv preprint arXiv:2006.01631*, 2020.

——,*Beliefs by proxy*, Some definitions have changed from then, and some results have been corrected., 2021. [Online]. Available: https://tsmithe.net/papers/2021-cybercat-bbp.pdf.

——,*Some notions of (open) dynamical system on polynomial interfaces*, 2021. arXiv: 2108.11137 [math.DS].

D. I. Spivak, "Learners' languages", *arXiv preprint arXiv:2103.01189*, 2021.