

Dependent lenses are dependent optics

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Introduction



12/21

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Based on Braithwaite et al. [2021](#); Vertech [2022](#); Capucci [2022](#).

Lenses

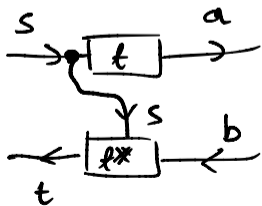
\mathcal{C} cartesian monoidal (\perp, \times)

$$f: S \xrightarrow{\text{read}} a$$

$$\underbrace{\begin{pmatrix} s \\ t \end{pmatrix} \begin{matrix} \longrightarrow & \longleftarrow \\ & \end{matrix} \begin{pmatrix} a \\ b \end{pmatrix}}$$

$$f^*: S \times b \xrightarrow{\text{write}} t$$

$$\text{Lens}(\mathcal{C})\left(\begin{pmatrix} s \\ t \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right)$$



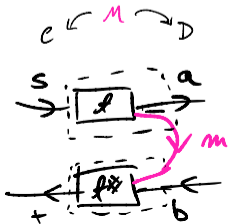
Mixed optics and F -lenses

See Román 2020; Clarke et al. 2020 and Spivak 2019.

mixed optics

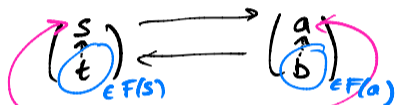
$$\begin{pmatrix} s \\ t \end{pmatrix} \begin{matrix} \longrightarrow \\ \longleftarrow \end{matrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{aligned} f: s &\longrightarrow m \cdot a \\ f^*: t &\longleftarrow m \cdot b \end{aligned}$$



F -lenses

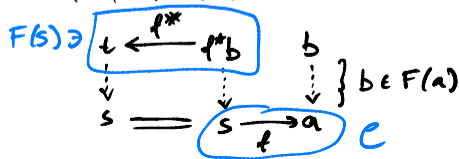
\approx Grothendieck comst



$$F: \mathcal{C}^{\text{op}} \longrightarrow \text{Cat}$$

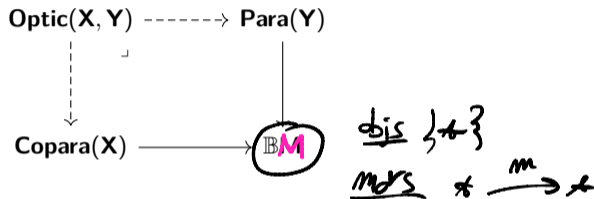
$$f: s \longrightarrow a \text{ in } \mathcal{C}$$

$$f^*: (b: s) \times b(x: a) \longrightarrow t: a$$

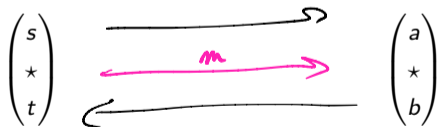


The Para/Copara decomposition

(ops and cos omitted for sanity):



which in turn suggests the following picture:



Also works for dependent lenses, which are coproducts of lenses, so can be considered 'indexed optics'.

Ansatz: actions of bicategories

Ansatz: F -lenses and mixed optics are special cases of this construction for generic actions of *bicategories*, for which we'd have

$$\begin{array}{ccc} \mathbf{Optic}(X, Y) & \dashrightarrow & \mathbf{Para}(Y) \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{Copara}(X) & \longrightarrow & \mathbb{M} \end{array}$$

What are actions of bicategories? What do we get?

Actions of categories

Action of a monoid M on a set X is a function (satisfying some properties):

$$\begin{array}{l} M \times X \longrightarrow X \\ (m, x) \longmapsto m \circ x \end{array}$$

Equivalently, it's a functor:

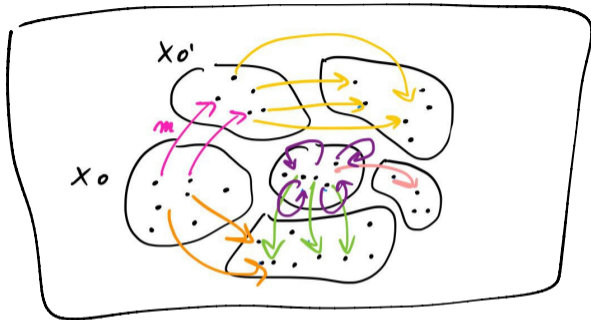
$$\begin{array}{ccc} BM & \longrightarrow & \text{Set} \\ \star & & X \\ m \downarrow & \longmapsto & \downarrow m \circ - \\ \star & & X \end{array}$$

If the monoid 'has many objects', we get the *action of a category* \mathbf{M} on a set X :

$$\begin{array}{ccc} \mathbf{M} & \longrightarrow & \text{Set} \\ o & & X(o) \\ f \downarrow & \longmapsto & \downarrow f \circ - \\ o' & & X(o') \end{array}$$

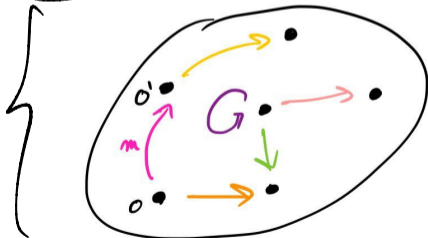
Actions of categories

Set



$$X = \sum_{o \in M_0} X(o)$$

Category **M**



Actions of bicategories

Let's categorify!

The action of a monoidal category \mathbf{M} on a category \mathbf{X} is a functor (plus some extra structure, see Capucci and Gavranović 2022):

$$\mathbf{M} \times \mathbf{X} \longrightarrow \mathbf{X}$$

Equivalently, it's a pseudofunctor:

$$\mathbf{M} \times \mathbf{M} \xrightarrow{\mathbb{B}\mathbf{M}} \mathbf{M} \xrightarrow{\mathbf{G}} \mathbf{X} \xrightarrow{\mathbf{m}} \mathbf{X}$$

$$\begin{array}{c} \mathbb{B}\mathbf{M} \\ \downarrow \mathbf{m} \\ \mathbf{M} \end{array} \xrightarrow{\mathbf{G}} \begin{array}{c} \mathbf{M} \\ \downarrow \mathbf{m} \\ \mathbf{X} \end{array}$$

If the monoidal category 'has many objects', we get the *action of a bicategory* \mathbf{M} on a ~~cat~~ \mathbf{X} :

$$\mathbf{M} \xrightarrow{\mathbf{M}} \mathbf{Cat} \quad \begin{array}{c} \mathbf{X}(0) \\ \downarrow \Rightarrow \downarrow \\ \mathbf{X}(0') \end{array}$$

(It turns out, if you're interested about Tambara theory it's really helpful to think about these as *doubly indexed categories* instead, but I digress...)

Dependent optics

Definition

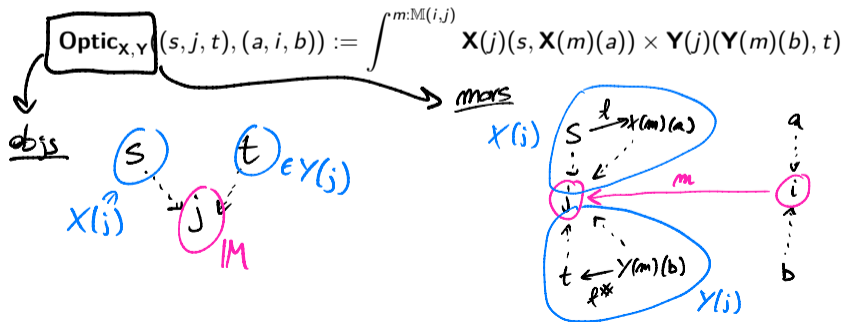
Let

\mathbb{M} be a bicategory,

$\mathbf{X} : \mathbb{M} \rightarrow \mathbf{Cat}$ be a pseudofunctor,

$\mathbf{Y} : \mathbb{M} \rightarrow \mathbf{Cat}$ be a pseudofunctor.

Then define



Degenerate dependent optics

$$X = C \quad \text{forward}$$

$$Y = D \quad \text{backward}$$

Example (Mixed optics)

If $\mathbb{M} = \mathbb{B}\mathbf{M}$ for a monoidal category \mathbf{M} (i.e. \mathbb{M} is 'trivial in the objects'), then

$$\begin{aligned} \text{Optic}_{X,Y}((s, *, t), (a, *, b)) &= \int^{m: \mathbb{B}\mathbf{M}(*, *)} \underbrace{X(*)(s, X(m)(a))}_{C} \times \underbrace{Y(*)(Y(m)(b), t)}_{D} \\ &= \int^{m: \mathbf{M}} X(s, m \bullet a) \times Y(m \circ b, t) \end{aligned}$$

where $m \bullet - := X(m)$ $m \circ - := Y(m)$.

Degenerate dependent optics

Example (F -lenses)

If $\mathbb{M} = \mathbf{I}^{\text{op}}$, i.e. it is 'trivial in the 2-cells', and \mathbf{X} is trivial, then

$$\text{Optic}_{1, \mathbf{Y}}((*)j, t), (*i, b) = \int_{m: \mathbf{I}^{\text{op}}(i, j)}^{\text{set}} 1 \times \mathbf{Y}(j)(\mathbf{Y}(m)(b), t)$$

the role is played by the residual \rightarrow no general part!

$$= \sum_{m: \mathbf{I}(j, i)} \mathbf{Y}(j)(m^* b, t)$$

Here $F = \mathbf{Y} : \mathbf{I}^{\text{op}} \rightarrow \mathbf{Cat}$ and $m^* = \mathbf{Y}(m) : \mathbf{Y}(i) \rightarrow \mathbf{Y}(j)$.

Dependent lenses

$$F = \mathbb{I}/- : \mathbb{I}^{\mathcal{Q}} \rightarrow \mathbf{Cat}$$
$$\begin{array}{ccc} i & & \mathbb{I}/i \\ \ell \downarrow & \mapsto & \uparrow \ell^* \\ j & & \mathbb{I}/j \end{array}$$

In a sense, we already recovered dependent lenses.

Taking \mathbb{I} locally cartesian, and one gets

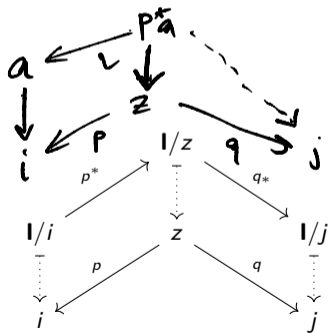
$$\mathbf{DLens}((s \leftarrow t), (a \leftarrow b)) = \mathbf{Optic}_{\mathbb{I}/-}((*, s, t), (*, a, b))$$

We can do better though! (as observed by P.V.,
we can avoid to throw away
the forward part so we can
use the residual to store information)

Dependent locally cartesian optics

Consider a span $i \xleftarrow{p} z \xrightarrow{q} j$ in \mathbf{I} .

This defines a functor:

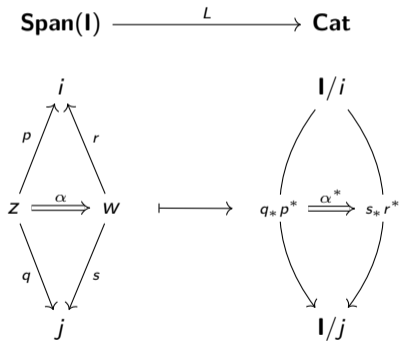


pull-push
operation

Dependent locally cartesian optics

I finite limits

In turn, we get an action of the bicategory of spans:



linear
polynomials
(see Gambino-Kock
and Schreiber 2014)

Dependent locally cartesian optics

And we compute...

$$\begin{aligned} \mathbf{Optic}_{L,L}(s \xrightarrow{v} j \xleftarrow{v'} t, a \xrightarrow{u} i \xleftarrow{u'} b) &:= \int^{i \xleftarrow{p} z \xrightarrow{q} j} \mathbf{I}/j(v, q_* p^* u) \times \mathbf{I}/j(q_* p^* u', v') \\ &\cong \int^{\langle p, q \rangle : z \rightarrow i \times j} \sum_{f \in \mathbf{I}(s, a)} \mathbf{I}/(i \times j)(\langle f \circ u, v \rangle, \langle p, q \rangle) \times \mathbf{I}/j(q_* p^* u', v') \\ &\cong \sum_{f \in \mathbf{I}(s, a)} \int^{\langle p, q \rangle : z \rightarrow i \times j} \mathbf{I}/(i \times j)(\langle f \circ u, v \rangle, \langle p, q \rangle) \times \mathbf{I}/j(q_* p^* u', v') \\ &\cong \sum_{f \in \mathbf{I}(s, a)} \mathbf{I}/j(v_*(f \circ u)^* u', v') \end{aligned}$$

(Lemma 4.1 in Capucci 2022)

Dependent locally cartesian optics

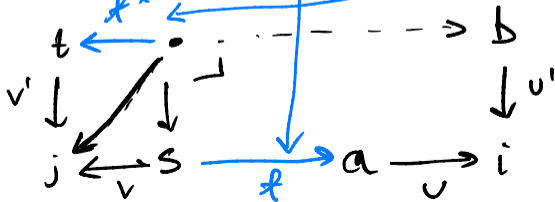
The category of dependent locally cartesian optics:

objects:

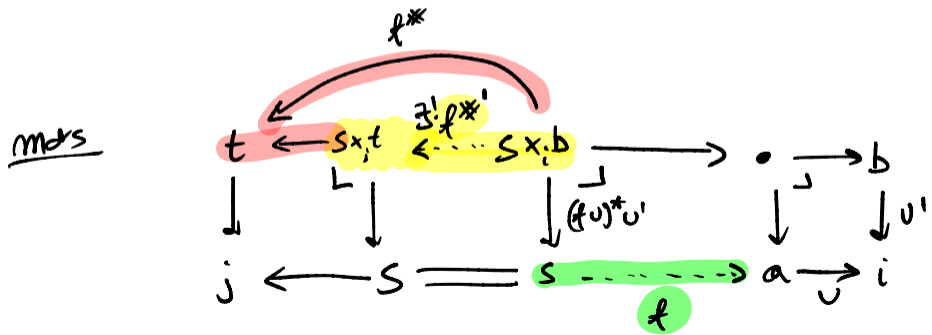
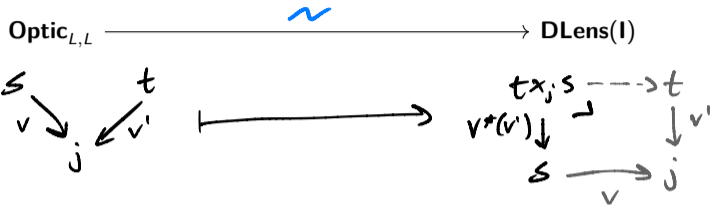


morphisms:

$$\mathbf{Optic}_{L,L}(s \xrightarrow{v} j \xleftarrow{v'} t, a \xrightarrow{u} i \xleftarrow{u'} b) \cong \sum_{f \in \mathbf{I}(s,a)} \mathbf{I}/j(v_*(f \circ u)^* u', v')$$



Equivalence to dependent lenses



Conclusions & future directions

Are locally cartesian dependent (2-)optics *operationally* different than dependent lenses?
Lenses and cartesian (2-)optics have different operational properties.







There seems to be an adjunction between **Dial** and **Optic**_{L,L} (and, therefore, **Poly**)

Work out more examples of *dependent optics*!

Thanks for your attention!

Questions?

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