On a fibrational construction for optics, lenses, and Dialectica categories

Matteo Capucci[1]

Jonathan Weinberger[2]

Bruno Gavranović[1]

Francisco Rios[3]

Abdullah Malik[4]

(1) University of Strathclyde

(2) Johns Hopkins University

(3) Quantum Information Science Section
Oak Ridge National Laboratory

(4) Florida State University

AMS Special Session on Applied Category Theory

Boston (USA), January 5th 2023

Acknowledgments

Thanks AMS for the great opportunity!

Acknowledgments

Thanks AMS for the great opportunity!

Acknowledgments

Thanks AMS for the great opportunity!

Introduction

Lenses, optics, and morphisms in Dialectica categories describe ‘dialogues’, bidirectional transformations between two sides sending back and forth messages.

Introduction

Lenses, optics, and morphisms in Dialectica categories describe ‘dialogues’, bidirectional transformations between two sides sending back and forth messages.

Hence the question we started to ask is: **how are they related?**

Introduction

Lenses, optics, and morphisms in Dialectica categories describe ‘dialogues’, bidirectional transformations between two sides sending back and forth messages.

Hence the question we started to ask is: **how are they related?**

This talk will attempt to answer that question.

References

- P. Hofstra, "The dialectica monad and its cousins", *Models, logics, and higherdimensional categories: A tribute to the work of Mihály Makkai*, vol. 53, pp. 107–139, 2011.
- S. K. Moss and T. von Glehn, "Dialectica models of type theory", in *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, arXiv:2105.00283 [cs, math], Jul. 2018, pp. 739–748. DOI: 10.1145/3209108.3209207. [Online]. Available: http://arxiv.org/abs/2105.00283 (visited on 01/04/2023).
- D. J. Myers, *Cartesian Factorization Systems and Grothendieck Fibrations*, en, arXiv:2006.14022 [math], Jan. 2021. [Online]. Available: http://arxiv.org/abs/2006.14022 (visited on 01/01/2023).
- V. de Paiva and J. Gray, "The dialectica categories", *Categories in Computer Science and Logic*, vol. 92, pp. 47–62, 1989, Publisher: Contemp. Math.
- D. I. Spivak, "Generalized lens categories", arXiv preprint available at https://arxiv.org/1908.02202, 2019.

Fun with fibrations

Lenses

Let $\mathbf{C}$ be cartesian monoidal.

Definition

The category of lenses $\mathbf{Lens}(\mathbf{C})$ has

- as objects, pairs (X, U) of objects of $\mathbf{C}$,
- as morphisms $(X, U) \leftrightarrows (Y, V)$, pairs

$$
\begin{aligned}
&f : U \to V, \\
&f^\sharp : U \times Y \to X
\end{aligned}
$$

Lenses

Let $\mathbf{C}$ be cartesian monoidal.

Definition

The category of lenses $\mathbf{Lens}(\mathbf{C})$ has

- as objects, pairs (X, U) of objects of $\mathbf{C}$,
- as morphisms $(X, U) \leftrightarrows (Y, V)$, pairs

$$f : U \to V,$$
$$f^\sharp : U \times Y \to X$$

A lens represents a back and forth dialogue: it answers to question coming from its left boundaries by asking questions to its right boundary.

Lenses

$$
\begin{array}{ccccc}
U \times X & \xleftarrow{f^\sharp} & U \times Y & \xrightarrow{f \times Y} & V \times Y \\
\downarrow{\scriptstyle \pi_U} & & \downarrow{\scriptstyle \pi_U} \;\lrcorner & & \downarrow{\scriptstyle \pi_V} \\
U & = & U & \xrightarrow{f} & V
\end{array}
$$

Dependent lenses

Let $\mathbf{C}$ be finitely complete

Definition

The category of dependent lenses $\mathbf{DLens}(\mathbf{C})$ has

- as objects, bundles $\begin{matrix} X \\ \downarrow \\ U \end{matrix}$ (i.e. morphisms) in $\mathbf{C}$,
- as morphisms $\begin{matrix} X \\ \downarrow \\ U \end{matrix} \leftrightarrows \begin{matrix} Y \\ \downarrow \\ V \end{matrix}$, pairs

$$f : U \to V,$$

$$f^\sharp : (u : U) \times Y(f(u)) \to X(u)$$

Dependent lenses

Let $\mathbf{C}$ be finitely complete

Definition

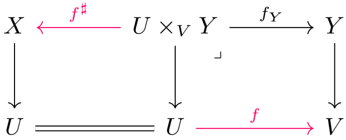
The category of dependent lenses $\mathbf{DLens}(\mathbf{C})$ has

- as objects, bundles $\begin{array}{c} X \\ \downarrow \\ U \end{array}$ (i.e. morphisms) in $\mathbf{C}$,
- as morphisms $\begin{array}{c} X \\ \downarrow \\ U \end{array} \leftrightarrows \begin{array}{c} Y \\ \downarrow \\ V \end{array}$, pairs

$$f : U \to V,$$
$$f^\sharp : (u : U) \times Y(f(u)) \to X(u)$$

A dependent lens represents a back and forth dialogue *with strict rules on which type of answers we are allowed to give for a question*.

Dependent lenses



p-lenses

Let $p : \mathbf{E} \to \mathbf{C}$ be a fibration/let $p^{-1} : \mathbf{C}^{\mathsf{op}} \to \mathbf{Cat}$ be an indexed category.[1]

Definition

The category of p-lenses has

- as objects, p-bundles $\begin{pmatrix} X : p^{-1} U \\ U : \mathbf{C} \end{pmatrix}$ (i.e. objects of $\mathbf{E}$)
- as morphisms $\begin{pmatrix} X \\ U \end{pmatrix} \leftrightarrows \begin{pmatrix} Y \\ V \end{pmatrix}$, morphisms:

$$\begin{aligned} f : U \to V \quad & : \mathbf{C} \\ f^\sharp : f^* Y \to X \quad & : p^{-1} U \end{aligned}$$

[1] To me every fibration is effectively cloven.

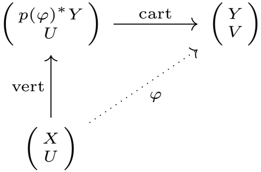
All lenses are p-lenses

	data
lenses	$\mathrm{cod} : \mathbf{C}^{\downarrow}_{\mathrm{proj}} \longrightarrow \mathbf{C}$
dependent lenses	$\mathrm{cod} : \mathbf{C}^{\downarrow} \longrightarrow \mathbf{C}$
p**-lenses**	$p : \mathbf{E} \longrightarrow \mathbf{C}$

Fibrations & vertical-cartesian factorization system

Definition

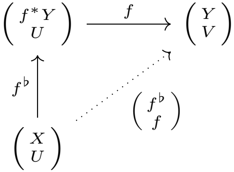
Any Grothendieck fibration $p : \mathbf{E} \to \mathbf{C}$ induces a factorization systems on $\mathbf{E}$ where the left morphisms are vertical morphisms $(p(f) = 1)$ and the right morphisms are cartesian.



Fibrations & vertical-cartesian factorization system

Definition

Any Grothendieck fibration $p : \mathbf{E} \to \mathbf{C}$ induces a factorization systems on $\mathbf{E}$ where the left morphisms are vertical morphisms ($p(f) = 1$) and the right morphisms are cartesian.



Idea: a fibred category is made of morphisms from the fibers (vertical morphisms) composed with morphisms from the base (cartesian).

p-lenses from dual fibrations

Definition

Given a Grothendieck fibration $p : \mathbf{E} \to \mathbf{C}$, we can form its **dual** or **(fiberwise) opposite**

$$p^\vee : \mathbf{E}^\vee \longrightarrow \mathbf{C}$$

obtained by replacing each fiber with its opposite: $p^\vee = \int (p^{-1} \mathbin{⨟} (-)^{\mathsf{op}})$.

p-lenses from dual fibrations

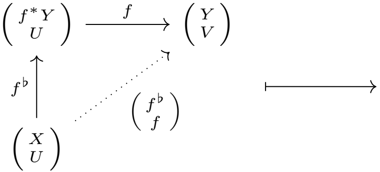
Definition

Given a Grothendieck fibration $p : \mathbf{E} \to \mathbf{C}$, we can form its **dual** or **(fiberwise) opposite**

$$p^\vee : \mathbf{E}^\vee \longrightarrow \mathbf{C}$$

obtained by replacing each fiber with its opposite: $p^\vee = \int (p^{-1} \mathbin{;} (-)^{\mathsf{op}})$.

$\mathbf{E}^\vee$ has the same objects but now morphisms are constructed from a cartesian and vertical part *pointing in opposite directions*:



p-lenses from dual fibrations

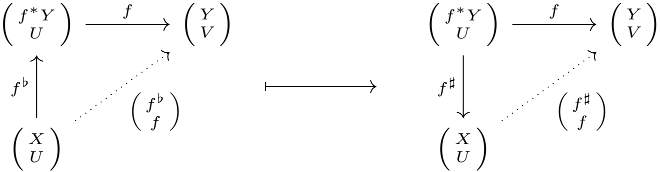
Definition

Given a Grothendieck fibration $p : \mathbf{E} \to \mathbf{C}$, we can form its **dual** or **(fiberwise) opposite**

$$p^{\vee} : \mathbf{E}^{\vee} \longrightarrow \mathbf{C}$$

obtained by replacing each fiber with its opposite: $p^{\vee} = \int (p^{-1} \mathbin{⨟} (-)^{\mathsf{op}})$.

$\mathbf{E}^{\vee}$ has the same objects but now morphisms are constructed from a cartesian and vertical part *pointing in opposite directions*:



This is a p-lens!

Dialectica

Let $\mathbf{C}$ be cartesian monoidal.

Definition

The Dialectica category $D\mathbf{C}$ has

Dialectica

Let $\mathbf{C}$ be cartesian monoidal.

Definition

The Dialectica category $D\mathbf{C}$ has

- as objects, triples

$$(U : \mathbf{C}, X : \mathbf{C}, \alpha : \mathrm{Sub}(U \times X))$$

Dialectica

Let $\mathbf{C}$ be cartesian monoidal.

Definition

The Dialectica category $D\mathbf{C}$ has

- as objects, triples

$$(U : \mathbf{C}, X : \mathbf{C}, \alpha : \mathrm{Sub}(U \times X))$$

- as morphisms $(U, X, \alpha) \to (V, Y, \beta)$, triples

$$f : U \to V,$$

$$f^\sharp : U \times Y \to X,$$

$$\forall u : U, y : Y, \quad \alpha(u, f^\sharp(u, y)) \subseteq \beta(f(u), y)$$

Dialectica

Let $\mathbf{C}$ be cartesian monoidal.

Definition

The Dialectica category $D\mathbf{C}$ has

- as objects, triples
$$(U : \mathbf{C}, X : \mathbf{C}, \alpha : \mathrm{Sub}(U \times X))$$
- as morphisms $(U, X, \alpha) \to (V, Y, \beta)$, triples
$$f : U \to V,$$
$$f^\sharp : U \times Y \to X,$$
$$\forall u : U, y : Y, \quad \alpha(u, f^\sharp(u, y)) \subseteq \beta(f(u), y)$$

Hence $D\mathbf{C}$ is a category of lenses 'augmented with predicates'.

Dialectica

Let $\mathbf{C}$ be cartesian monoidal.

Definition

The Dialectica category $D\mathbf{C}$ has

- as objects, triples

$$(U : \mathbf{C}, X : \mathbf{C}, \alpha : \mathrm{Sub}(U \times X))$$

- as morphisms $(U, X, \alpha) \to (V, Y, \beta)$, triples

$$\begin{aligned} & f : U \to V, \\ & f^\sharp : U \times Y \to X, \\ & \forall u : U, y : Y, \quad \alpha(u, f^\sharp(u, y)) \subseteq \beta(f(u), y) \end{aligned}$$

Hence $D\mathbf{C}$ is a category of lenses 'augmented with predicates'.

Dialectica

Let $\mathbf{C}$ be cartesian monoidal.

Definition

The Dialectica category $D\mathbf{C}$ has

- as objects, triples

$$(U : \mathbf{C}, X : \mathbf{C}, \alpha : \mathrm{Sub}(U \times X))$$

- as morphisms $(U, X, \alpha) \to (V, Y, \beta)$, triples

$$f : U \to V,$$
$$f^\sharp : U \times Y \to X,$$
$$\forall u : U, y : Y, \quad \alpha(u, f^\sharp(u, y)) \subseteq \beta(f(u), y)$$

Hence $D\mathbf{C}$ is a category of lenses 'augmented with predicates'.

Can $D\mathbf{C}$ be constructed in a similar way to $\mathbf{Lens}(\mathbf{C})$?

Dialectica

Let $\mathbf{C}$ be cartesian monoidal.

Definition

The Dialectica category $D\mathbf{C}$ has

- as objects, triples

$$(U : \mathbf{C}, X : \mathbf{C}, \alpha : \mathrm{Sub}(U \times X))$$

- as morphisms $(U, X, \alpha) \to (V, Y, \beta)$, triples

$$\begin{aligned} &f : U \to V, \\ &f^\sharp : U \times Y \to X, \\ &\forall u : U, y : Y, \quad \alpha(u, f^\sharp(u, y)) \subseteq \beta(f(u), y) \end{aligned}$$

Hence $D\mathbf{C}$ is a category of lenses 'augmented with predicates'.

Can $D\mathbf{C}$ be constructed in a similar way to $\mathbf{Lens}(\mathbf{C})$?

Can its shape be abstracted, like we did for p-lenses?

Dialectica, fibrationally

Suppose we replace the lens part with a p-lens, where do we get the predicates?

Dialectica, fibrationally

Suppose we replace the lens part with a p-lens, where do we get the predicates?

We need a second fibration of predicates!

$$
\begin{array}{cc}
\mathbf{P} & \text{`}\alpha \subseteq \beta\text{'} \\
\big\downarrow{\scriptstyle q} & \\
\mathbf{E} & f^\sharp : f^*Y \to X \\
\big\downarrow{\scriptstyle p} & \\
\mathbf{C} & f : U \to V
\end{array}
$$

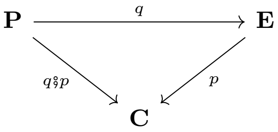
Dialectica, fibrationally

What happens to q when we use p to produce $p^{\vee}$, the fibration of lenses?

Dialectica, fibrationally

What happens to q when we use p to produce $p^\vee$, the fibration of lenses?

First, notice:



$$p ⨾ q \xrightarrow{q} p : \mathbf{Fib}(\mathbf{C})$$

which means the triangle commutes & that q respects cartesian arrows (both trivial).

Dialectica, fibrationally

What happens to q when we use p to produce $p^\vee$, the fibration of lenses?

First, notice:

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{\quad q \quad} & \mathbf{E} \\ & {\scriptstyle q ⨾ p} \searrow \quad \swarrow {\scriptstyle p} & \\ & \mathbf{C} & \end{array} \qquad p ⨾ q \xrightarrow{q} p : \mathbf{Fib}(\mathbf{C})$$

which means the triangle commutes & that q respects cartesian arrows (both trivial).

Then we notice $(-)^\vee$ is a functorial construction $\mathbf{Fib}(\mathbf{C}) \to \mathbf{Fib}(\mathbf{C})$, hence can be applied to the whole triangle:

$$\begin{array}{ccc} \mathbf{P}^\vee & \xrightarrow{\quad q^\vee \quad} & \mathbf{E}^\vee \\ & {\scriptstyle q ⨾ p^\vee} \searrow \quad \swarrow {\scriptstyle p^\vee} & \\ & \mathbf{C} & \end{array}$$

Is this what we look for? Let's unpack.

Dialectica, fibrationally

Remember $\mathbf{E}$ has a factorization system which $\mathbf{E}^{\vee}$ inherits as **(vertical$^{\text{op}}$, cartesian)**.

Dialectica, fibrationally

Remember $\mathbf{E}$ has a factorization system which $\mathbf{E}^\vee$ inherits as **(vertical$^{\text{op}}$, cartesian)**.

On $\mathbf{P}$ we had a more refined factorization system, since we have three kinds of arrows:

- q-**cartesian arrows**, which are cartesian lifts of $\mathbf{E}$-arrows, and come in two subcategories:
  - p-**cartesian arrows**, which are cartesian lifts of $\mathbf{C}$-arrows, hence cartesian $\mathbf{E}$-arrows,
  - p-**vertical arrows**, which are cartesian lifts of vertical $\mathbf{E}$-arrows,
- q-**vertical arrows**, which are in the fibers of q.

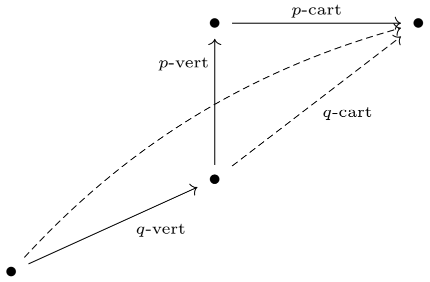
Dialectica, fibrationally

Remember $\mathbf{E}$ has a factorization system which $\mathbf{E}^\vee$ inherits as **(vertical$^{\mathsf{op}}$, cartesian)**.

On $\mathbf{P}$ we had a more refined factorization system, since we have three kinds of arrows:

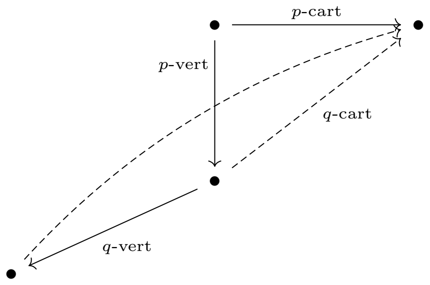
- q-**cartesian arrows**, which are cartesian lifts of $\mathbf{E}$-arrows, and come in two subcategories:
  - p-**cartesian arrows**, which are cartesian lifts of $\mathbf{C}$-arrows, hence cartesian $\mathbf{E}$-arrows,
  - p-**vertical arrows**, which are cartesian lifts of vertical $\mathbf{E}$-arrows,
- q-**vertical arrows**, which are in the fibers of q.

This forms a ternary factorization system **(q-vertical, p-vertical, p-cartesian)**:



Dialectica, fibrationally

When we turn around the fibers of $\mathbf{E}$, we swap q-vertical and p-vertical arrows:



Hence on $\mathbf{P}^\vee$ we end up with a ternary factorization system where p- and q-vertical arrows are swapped:

$$(p\text{-}\textbf{vertical}^{\mathsf{op}},\ q\text{-}\textbf{vertical}^{\mathsf{op}},\ p\text{-}\textbf{cartesian})$$

Dialectica, fibrationally

We can understand this factorization system as arising from an **ambifibration** structure on $q^\vee$:

Definition

Let (L, R) be a factorization system on $\mathbf{D}$. An **ambifibration** $a : \mathbf{F} \to \mathbf{D}$ is a functor such that

- every arrow in L has an opcartesian lift (a is an opfibration on L)
- every arrow in R has a cartesian lift (a is a fibration on R)

This induces the ternary factorization system **(opcartesian, vertical, cartesian)** on $\mathbf{F}$:

$$\begin{array}{ccccccccc} \mathbf{F} & & A' & \xrightarrow[\text{opcart}]{} & \ell_* A' & \xrightarrow[\text{vert}]{} & r^* C' & \xrightarrow[\text{cart}]{} & C' \\ \downarrow a & & & & & & & & \\ \mathbf{D} & & A & \xrightarrow[\in L]{} & B & = & B & \xrightarrow[\in R]{} & C \end{array}$$

Dialectica, fibrationally

In our case:

- $\mathbf{E}^{\vee}$ has the factorization system **(vertical$^{\mathsf{op}}$, cartesian)**
- $q^{\vee}$ is a fibration on the cartesian class of $\mathbf{E}$, given by q, and became an opfibration on the vertical class because it acts like q^{op} there:

$$\begin{pmatrix}\alpha\\X\\U\end{pmatrix} \xleftarrow{\quad f^{\sharp}_{\alpha}\quad} \begin{pmatrix}(f^{\sharp})^{*}\alpha\\X\\U\end{pmatrix} \xleftarrow{\quad f^{\times}\quad} \begin{pmatrix}f^{*}\beta\\X\\U\end{pmatrix} \xrightarrow{\quad f_{\beta}\quad} \begin{pmatrix}\beta\\Y\\V\end{pmatrix}$$

$$\begin{pmatrix}X\\U\end{pmatrix} \xleftarrow{\quad f^{\sharp}\quad} \begin{pmatrix}f^{*}Y\\U\end{pmatrix} = \begin{pmatrix}f^{*}Y\\U\end{pmatrix} \xrightarrow{\quad f\quad} \begin{pmatrix}Y\\V\end{pmatrix}$$

Dialectica, fibrationally

In our case:

- $\mathbf{E}^{\vee}$ has the factorization system **(vertical$^{\mathsf{op}}$, cartesian)**
- $q^{\vee}$ is a fibration on the cartesian class of $\mathbf{E}$, given by q, and became an opfibration on the vertical class because it acts like q^{op} there:

$$\begin{pmatrix} \alpha \\ X \\ U \end{pmatrix} \xleftarrow{\;f^{\sharp}_{\alpha}\;} \begin{pmatrix} (f^{\sharp})^{*}\alpha \\ X \\ U \end{pmatrix} \xleftarrow{\;f^{\maltese}\;} \begin{pmatrix} f^{*}\beta \\ X \\ U \end{pmatrix} \xrightarrow{\;f_{\beta}\;} \begin{pmatrix} \beta \\ Y \\ V \end{pmatrix}$$

$$\begin{pmatrix} X \\ U \end{pmatrix} \xleftarrow{\;f^{\sharp}\;} \begin{pmatrix} f^{*}Y \\ U \end{pmatrix} = \begin{pmatrix} f^{*}Y \\ U \end{pmatrix} \xrightarrow{\;f\;} \begin{pmatrix} Y \\ V \end{pmatrix}$$

This is very close: we have f, $f^{\sharp}$ and the right boundaries for $f^{\maltese}$.

The iterated dual construction

Let $p = \mathbf{E}_n \xrightarrow{p_n} \cdots \xrightarrow{p_2} \mathbf{E}_1 \xrightarrow{p_1} \mathbf{E}_0$ be a sequence of n fibrations.

Definition

The **iterated dual construction** is defined inductively as follows:

- $n = 1$

$$\left(\mathbf{E}_1 \xrightarrow{p_1} \mathbf{E}_0\right)^{\vee_1} = \mathbf{E}_1{}^{\vee_{\mathbf{E}_0}} \xrightarrow{p_1{}^{\vee_{\mathbf{E}_0}}} \mathbf{E}_0$$

- $n = k + 1$

$$\left(\mathbf{E}_{k+1} \xrightarrow{p_{k+1}} \cdots \xrightarrow{p_2} \mathbf{E}_1 \xrightarrow{p_1} \mathbf{E}_0\right)^{\vee_{k+1}} = \left(\mathbf{E}_{k+1} \xrightarrow{p_{k+1}} \cdots \mathbf{E}_1\right)^{\vee_{\mathbf{E}_0}\vee_k} \xrightarrow{p_1{}^{\vee_{\mathbf{E}_0}}} \mathbf{E}_0$$

The iterated dual construction

Let $p = \mathbf{E}_n \xrightarrow{p_n} \cdots \xrightarrow{p_2} \mathbf{E}_1 \xrightarrow{p_1} \mathbf{E}_0$ be a sequence of n fibrations.

Definition

The **iterated dual construction** is defined inductively as follows:

- $n = 1$

$$\left(\mathbf{E}_1 \xrightarrow{p_1} \mathbf{E}_0\right)^{\vee 1} = \mathbf{E}_1{}^{\vee \mathbf{E}_0} \xrightarrow{p_1{}^{\vee \mathbf{E}_0}} \mathbf{E}_0$$

- $n = k + 1$

$$\left(\mathbf{E}_{k+1} \xrightarrow{p_{k+1}} \cdots \xrightarrow{p_2} \mathbf{E}_1 \xrightarrow{p_1} \mathbf{E}_0\right)^{\vee k+1} = \left(\mathbf{E}_{k+1} \xrightarrow{p_{k+1}} \cdots \mathbf{E}_1\right)^{\vee \mathbf{E}_0 \vee k} \xrightarrow{p_1{}^{\vee \mathbf{E}_0}} \mathbf{E}_0$$

Example $\left(\mathbf{E}_2 \xrightarrow{p_2} \mathbf{E}_1 \xrightarrow{p_1} \mathbf{E}_0\right)^{\vee 2} = \left(\mathbf{E}_2{}^{\vee \mathbf{E}_0}\right)^{\vee \mathbf{E}_1} \xrightarrow{\left(p_2{}^{\vee \mathbf{E}_0}\right)^{\vee \mathbf{E}_1}} \mathbf{E}_1{}^{\vee \mathbf{E}_0} \xrightarrow{p_1{}^{\vee \mathbf{E}_0}} \mathbf{E}_0$

The iterated dual construction

Let $p = \mathbf{E}_n \xrightarrow{p_n} \cdots \xrightarrow{p_2} \mathbf{E}_1 \xrightarrow{p_1} \mathbf{E}_0$ be a sequence of n fibrations.

Definition

The **iterated dual construction** is defined inductively as follows:

- $n = 1$

$$\left(\mathbf{E}_1 \xrightarrow{p_1} \mathbf{E}_0\right)^{\vee 1} = \mathbf{E}_1{}^{\vee \mathbf{E}_0} \xrightarrow{p_1{}^{\vee \mathbf{E}_0}} \mathbf{E}_0$$

- $n = k + 1$

$$\left(\mathbf{E}_{k+1} \xrightarrow{p_{k+1}} \cdots \xrightarrow{p_2} \mathbf{E}_1 \xrightarrow{p_1} \mathbf{E}_0\right)^{\vee k+1} = \left(\mathbf{E}_{k+1} \xrightarrow{p_{k+1}} \cdots \mathbf{E}_1\right)^{\vee \mathbf{E}_0 \vee k} \xrightarrow{p_1{}^{\vee \mathbf{E}_0}} \mathbf{E}_0$$

Example $\left(\mathbf{E}_2 \xrightarrow{p_2} \mathbf{E}_1 \xrightarrow{p_1} \mathbf{E}_0\right)^{\vee 2} = \left(\mathbf{E}_2{}^{\vee \mathbf{E}_0}\right)^{\vee \mathbf{E}_1} \xrightarrow{\left(p_2{}^{\vee \mathbf{E}_0}\right)^{\vee \mathbf{E}_1}} \mathbf{E}_1{}^{\vee \mathbf{E}_0} \xrightarrow{p_1{}^{\vee \mathbf{E}_0}} \mathbf{E}_0$

Definition

A morphism in $\mathbf{E}_n{}^{\vee n}$ is called a p-**dialens of height** n.

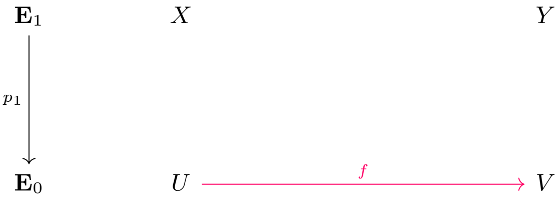
Dialens

Let's unpack the construction of a dialens of height 2:

$$\mathbf{E}_0 \qquad U \xrightarrow{\quad f \quad} V$$

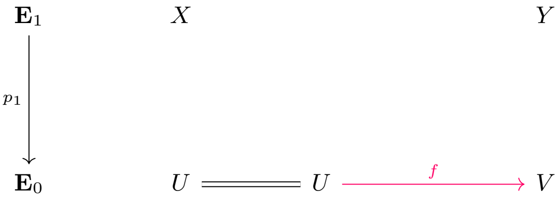
Dialens

Let's unpack the construction of a dialens of height 2:



Dialens

Let's unpack the construction of a dialens of height 2:



Dialens

Let's unpack the construction of a dialens of height 2:

$$
\begin{array}{cccccc}
\mathbf{E}_1 & & X & & f^*Y & \xrightarrow{\;f_Y\;} & Y \\
\Big\downarrow{\scriptstyle p_1} & & & & & & \\
\mathbf{E}_0 & & U & = & U & \xrightarrow{\;f\;} & V
\end{array}
$$

Dialens

Let's unpack the construction of a dialens of height 2:

$$
\begin{array}{cccccccc}
\mathbf{E}_1 & & X & \xrightarrow{f^\sharp} & f^*Y & \xrightarrow{\quad f_Y \quad} & Y \\
\big\downarrow {\scriptstyle p_1} & & & & & & \\
\mathbf{E}_0 & & U & = & U & \xrightarrow{\quad f \quad} & V
\end{array}
$$

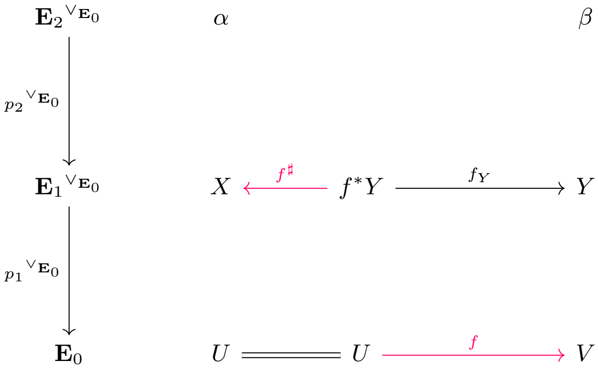
Dialens

Let's unpack the construction of a dialens of height 2:

$$\begin{array}{ccccccc} \mathbf{E}_1{}^{\vee \mathbf{E}_0} & \qquad & X & \xleftarrow{f^\sharp} & f^*Y & \xrightarrow{\quad f_Y \quad} & Y \\ \Big\downarrow{\scriptstyle p_1{}^{\vee \mathbf{E}_0}} & & & & & & \\ \mathbf{E}_0 & & U & = & U & \xrightarrow{\quad f \quad} & V \end{array}$$

Dialens

Let's unpack the construction of a dialens of height 2:



Dialens

Let's unpack the construction of a dialens of height 2:

$$
\begin{array}{cccccccc}
\mathbf{E}_2{}^{\vee \mathbf{E}_0} & & \alpha & & & & & \beta \\
\Big\downarrow {\scriptstyle p_2{}^{\vee \mathbf{E}_0}} & & & & & & & \\
\mathbf{E}_1{}^{\vee \mathbf{E}_0} & & X & \xleftarrow{f^\sharp} & f^*Y & = & f^*Y & \xrightarrow{f_Y} & Y \\
\Big\downarrow {\scriptstyle p_1{}^{\vee \mathbf{E}_0}} & & & & & & & \\
\mathbf{E}_0 & & U & = & U & = & U & \xrightarrow{f} & V
\end{array}
$$

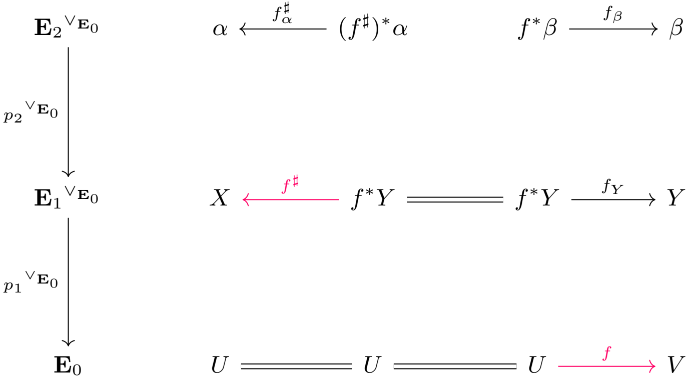
Dialens

Let's unpack the construction of a dialens of height 2:

$$
\begin{array}{ccccccccc}
\mathbf{E}_2{}^{\vee \mathbf{E}_0} & & \alpha & & & & f^*\beta & \xrightarrow{f_\beta} & \beta \\
\Big\downarrow {\scriptstyle p_2{}^{\vee \mathbf{E}_0}} & & & & & & & & \\
\mathbf{E}_1{}^{\vee \mathbf{E}_0} & & X & \xleftarrow{f^\sharp} & f^*Y & = & f^*Y & \xrightarrow{f_Y} & Y \\
\Big\downarrow {\scriptstyle p_1{}^{\vee \mathbf{E}_0}} & & & & & & & & \\
\mathbf{E}_0 & & U & = & U & = & U & \xrightarrow{f} & V
\end{array}
$$

Dialens

Let's unpack the construction of a dialens of height 2:



Dialens

Let's unpack the construction of a dialens of height 2:

$$
\begin{array}{ccccccccc}
\mathbf{E}_2{}^{\vee \mathbf{E}_0} & & \alpha & \xleftarrow{f^\sharp_\alpha} & (f^\sharp)^*\alpha & \xleftarrow{f^{\maltese}} & f^*\beta & \xrightarrow{f_\beta} & \beta \\
\downarrow {\scriptstyle p_2{}^{\vee \mathbf{E}_0}} & & & & & & & & \\
\mathbf{E}_1{}^{\vee \mathbf{E}_0} & & X & \xleftarrow{f^\sharp} & f^*Y & = & f^*Y & \xrightarrow{f_Y} & Y \\
\downarrow {\scriptstyle p_1{}^{\vee \mathbf{E}_0}} & & & & & & & & \\
\mathbf{E}_0 & & U & = & U & = & U & \xrightarrow{f} & V
\end{array}
$$

Dialens

Let's unpack the construction of a dialens of height 2:

$$\begin{array}{ccccccccc}
\mathbf{E}_2{}^{\vee\mathbf{E}_0}{}^{\vee\mathbf{E}_1} & \qquad & \alpha & \xleftarrow{\;f^\sharp_\alpha\;} & (f^\sharp)^*\alpha & \xrightarrow{\;f^{\times}\;} & f^*\beta & \xrightarrow{\;f_\beta\;} & \beta \\
{\scriptstyle p_2{}^{\vee\mathbf{E}_0}{}^{\vee\mathbf{E}_1}}\Big\downarrow & & & & & & & & \\
\mathbf{E}_1{}^{\vee\mathbf{E}_0} & & X & \xleftarrow{\;f^\sharp\;} & f^*Y & = & f^*Y & \xrightarrow{\;f_Y\;} & Y \\
{\scriptstyle p_1{}^{\vee\mathbf{E}_0}}\Big\downarrow & & & & & & & & \\
\mathbf{E}_0 & & U & = & U & = & U & \xrightarrow{\;f\;} & V
\end{array}$$


Genesis XXVIII c. 12 v

All things are p-dialenses

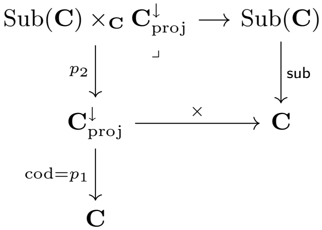
As expected, lenses are dialenses of height 1:

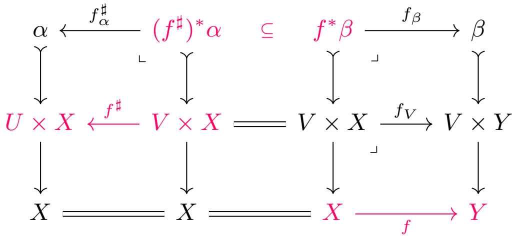
$$\begin{array}{lcccccc} & \mathbf{E} & U & \xleftarrow{f^\sharp} & f^*V & \xrightarrow{f_V} & V \\ p\textbf{-lenses} & \Big\downarrow{\scriptstyle p} & & & & & \\ & \mathbf{C} & X & = & X & \xrightarrow[f]{} & Y \end{array}$$

All things are p-dialenses

As expected, Dialectica morphisms are dialenses of height 2:

vanilla

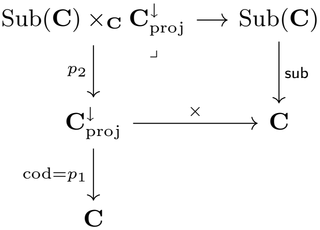


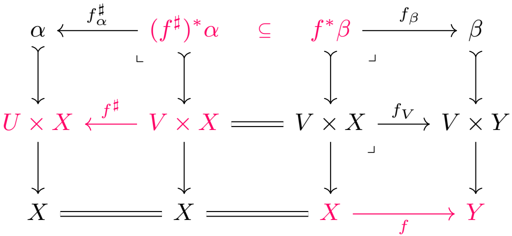


All things are p-dialenses

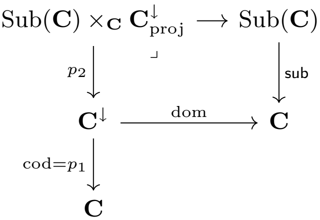
As expected, Dialectica morphisms are dialenses of height 2:

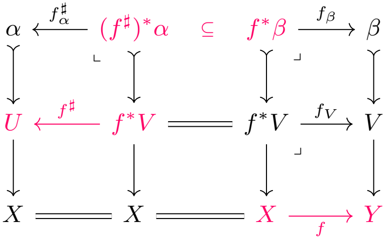
vanilla





dependent





All things are p-dialenses

Optics are p-dialenses of height 2 (but using opfibrations instead of fibrations!):

(op)tics

$$
\begin{array}{ccc}
\mathbf{Para}(\circ) \times_{\mathbf{B}M} \mathbf{Para}(\bullet) & \longrightarrow & \mathbf{Para}(\circ) \\
\downarrow {\scriptstyle p_2} & \lrcorner & \downarrow {\scriptstyle \int \circ} \\
\mathbf{Para}(\bullet) & \xrightarrow{\int \bullet} & \mathbf{B}M \\
\downarrow {\scriptstyle \int \bullet = p_1} & & \\
\mathbf{B}M & &
\end{array}
$$

$$
\begin{array}{l}
U \xrightarrow{m_U} m \circ U \xrightarrow{f^\sharp} V \overset{1_V}{=\!=\!=} V \\
X \xrightarrow{m_X} m \bullet X =\!=\!= m \bullet X \xleftarrow{f} Y \\
* \xrightarrow{m} * =\!=\!= * =\!=\!= *
\end{array}
$$

Comparison with Hofstra's $\mathfrak{Dial}$ monad

Hofstra defined a monad on fibrations that builds Dialectica-like categories by simple sum-product completion:

$$\mathfrak{Dial}(p) = \mathfrak{Sum}(\mathfrak{Prod}(p)) = \mathfrak{Sum}(\mathfrak{Sum}(p^\vee)^\vee)$$

where $p : \mathbf{P} \to \mathbf{C}$ is a fibration on $\mathbf{C}$ cartesian monoidal.

Comparison with Hofstra's $\mathfrak{Dial}$ monad

Hofstra defined a monad on fibrations that builds Dialectica-like categories by simple sum-product completion:

$$\mathfrak{Dial}(p) = \mathfrak{Sum}(\mathfrak{Prod}(p)) = \mathfrak{Sum}(\mathfrak{Sum}(p^\vee)^\vee)$$

where $p : \mathbf{P} \to \mathbf{C}$ is a fibration on $\mathbf{C}$ cartesian monoidal.

In $\mathfrak{Dial}(p)$, objects are triples $(I, X, U : \mathbf{C}, \alpha : \mathbf{P}(I \times X \times U))$ and morphisms have **four** parts

$$f_0 : I \to J$$

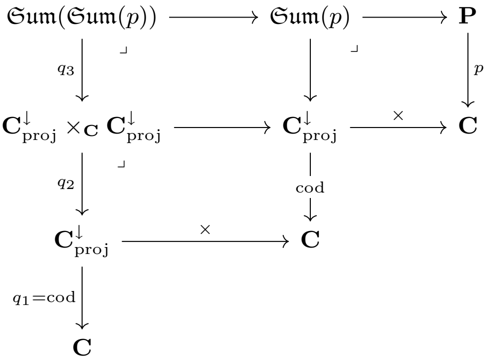
$$f : I \times X \to Y$$

$$f^\sharp : I \times X \times V \to U$$

given $i : I$, $x : X$, $v : V$, $f^{×} : \alpha(i, x, f^\sharp(i, x, v)) \to \beta(f_0(i), f(i, x), v)$

Comparison with Hofstra's $\mathfrak{Dial}$ monad

If we ignore the duals for a moment (they can be put back later), we see this is actually given by a sequence of **three** fibrations over $\mathbf{C}$:



Comparison with Hofstra's $\mathfrak{Dial}$ monad

If we ignore the duals for a moment (they can be put back later), we see this is actually given by a sequence of **three** fibrations over $\mathbf{C}$:

$$\begin{array}{ccccc}
\mathfrak{Sum}(\mathfrak{Sum}(p)) & \longrightarrow & \mathfrak{Sum}(p) & \longrightarrow & \mathbf{P} \\
{\scriptstyle q_3}\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\
\mathbf{C}^{\downarrow}_{\mathrm{proj}} \times_{\mathbf{C}} \mathbf{C}^{\downarrow}_{\mathrm{proj}} & \longrightarrow & \mathbf{C}^{\downarrow}_{\mathrm{proj}} & \xrightarrow{\times} & \mathbf{C} \\
{\scriptstyle q_2}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{cod}} & & \\
\mathbf{C}^{\downarrow}_{\mathrm{proj}} & \xrightarrow{\times} & \mathbf{C} & & \\
{\scriptstyle q_1=\mathrm{cod}}\downarrow & & & & \\
\mathbf{C} & & & &
\end{array}$$

One can see $\mathfrak{Dial}(p)$ is obtained by dualizing the top two:

$$\mathfrak{Dial}(p) = \left(\mathfrak{Sum}(\mathfrak{Sum}(p)) \xrightarrow{q_3} \mathbf{C}^{\downarrow}_{\mathrm{proj}} \times_{\mathbf{C}} \mathbf{C}^{\downarrow}_{\mathrm{proj}} \xrightarrow{q_2} \mathbf{C}^{\downarrow}_{\mathrm{proj}}\right)^{\vee_2} \xrightarrow{q_1} \mathbf{C}$$

Conclusions

Conclusions

In this talk we've seen

- how lenses arise from dual fibrations

In the future, we are going to

Conclusions

In this talk we've seen

- how lenses arise from dual fibrations
- how Dialectica categories are augmented categories of lenses

In the future, we are going to

Conclusions

In this talk we've seen

- how lenses arise from dual fibrations
- how Dialectica categories are augmented categories of lenses
- how this can be made precise by extending the lens construction to longer chains of fibrations

In the future, we are going to

Conclusions

In this talk we've seen

- how lenses arise from dual fibrations
- how Dialectica categories are augmented categories of lenses
- how this can be made precise by extending the lens construction to longer chains of fibrations

In the future, we are going to

- explore more examples of dialenses, for instance arising from 'differential structures',

Conclusions

In this talk we've seen

- how lenses arise from dual fibrations
- how Dialectica categories are augmented categories of lenses
- how this can be made precise by extending the lens construction to longer chains of fibrations

In the future, we are going to

- explore more examples of dialenses, for instance arising from 'differential structures',
- see if the structure of dialens is apt for dependent optics

Thanks for your attention!