Formal Observability and Behavioural Equivalence (Draft)

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Abstract

We study observability of systems in a completely formal setting, that is, without relying on specific notions of system. We do so by working at the level of theories of systems, i.e. algebraic structures that allow for the description of both the categorical and compositional nature of systems and their behaviours.

Specifically, we look at vertical and cartesian maps defined by a choice of behaviour of a theory of systems. We investigate when vertical–cartesian maps form a factorization system, which we define for theories of systems, and thus when they induce a reflective subtheory of 'fully observable systems'.

The behaviour–realization adjunction so constructed extends the classical observation that these are adjoint for automata and coalgebras. We give example that show how this generality elegantly ties together many different notions of observability.

1. Behaviours

Observability in general systems theory should be a notion defined in terms of the *observations* we make of a system, i.e. of a behaviour. A behaviour for a system theory is a map $B : \mathbf{Sys} \to \mathbf{Set}$. More generally, a behaviour can be any map $B : \mathbf{Sys} \to \mathbf{S}$ for $\mathbf{S}$ a 'behavioural theory', see [Mye22]; or even more generally any map of theories can be seen as giving semantics of its domain in the codomain. Given a behaviour $B : \mathbf{Sys} \to \mathbf{Set}$, there is class of maps of particular interest, that we denote as $\mathbf{ker}B$: these are all the B-vertical maps, i.e. the maps inverted by B. It's easy to see $\mathbf{ker}B$ is a full subtheory of $\mathbf{Sys}$, comprised of the same objects (i.e. interfaces, compositions and systems) but limited to those maps (i.e. maps of interfaces, squares and maps of systems) which are inverted by B.

Example 1.1. (Polynomial) coalgebraic behaviour can be described by

$$\mathbf{Lng} : \mathbf{Atm} \longrightarrow \mathbf{AtmLng} \tag{1.1}$$

where $\mathbf{Atm}$ is the theory of deterministic automata (coalgebras of polynomial functors with a distinguished initial state) while $\mathbf{AtmLng}$ is the theory with same interfaces as $\mathbf{Atm}$ (i.e. polynomial functors) but whose systems are confined to be subcoalgebras of the terminal coalgebra:

$$\mathbf{AtmLng}\binom{I}{O} = \left\{ L \subseteq \nu O \times (-)^I \right\}. \tag{1.2}$$

The functor **Lng** is identity on interfaces but sends an automaton $\mathsf{S} = (S, s_0, \delta)$ to the (subcoalgebra generated by the) language of its initial state. Notice that if S and R are both coalgebras over the same interface, and $\varphi : \mathsf{S} \to \mathsf{R}$ is a morphism, since the latter respects both initial state and dynamics it follows that $\mathbf{Lng}(\varphi)$ is the identity on the languages of (the initial states of) S and R. It follows that a general map of systems is a behavioural equivalence as soon as its component on interfaces (which remains unchanged under **Lng**) is an isomorphism.

Example 1.2. Things get more interesting once we allow for non-determinism. Let $\mathbf{Atm}^M$ be the theory of non-deterministic automata given by polynomial functors with M-effects, M being a commutative monad (e.g. P_ω, or D finite distributions monad). We still have a language functor:

$$\mathbf{Lng} : \mathbf{Atm}^M \longrightarrow \mathbf{AtmLng}^M \tag{1.3}$$

For $M =$ powerset monad P, $\mathbf{AtmLng}^P\left(\begin{smallmatrix} I \\ O \end{smallmatrix}\right) = \left\{L \subseteq O^{I^*}\right\}$ with maps given by inclusion, while for general M one just has a discrete category $\mathbf{AtmLng}^P\left(\begin{smallmatrix} I \\ O \end{smallmatrix}\right) = M(O^{I^*})$.[1]

The same argument as above shows that a map of M-non-deterministic automata with invertible component on the interfaces is always vertical. However, for $M = P$ one can weaken the definition of map so as to allow a map to only laxly preserve transitions. That is, the set of possible transitions at $s \in S$ can be strictly contained in the set of transitions available at $\varphi(s)$. This breaks the triviality of maps.

Example 1.3. If we don't fix an initial state, both maps of deterministic and non-deterministic automata are **Lng**-vertical (i.e. trace equivalences) iff their carrier is surjective (as well as having an invertible component on interfaces).

The reason the maps in $\mathbf{ker}B$ are important is that they classify obstructions to observability. Specifically, if $\mathbf{ker}B = \mathbf{Sys}^{\tilde{\to}}$, where the latter denotes the restriction of **Sys** to its invertible maps (its 'core'), then we could claim every system in **Sys** is fully B-observable: the only way two systems have indistinguishable behaviour is if they are equivalent to begin with.

Conversely, if $\mathbf{ker}B \supsetneq \mathbf{Sys}^{\tilde{\to}}$ then the non-invertible morphisms in $\mathbf{ker}B$ witness the fact that some systems have the same behaviour despite not being isomorphic.

If **Sys** supports an epi-mono factorization system, then we can divide the failures of $\mathbf{ker}B$ to be trivial in two classes:

1. $\mathbf{Sys}^{\rightarrowtail} \cap \mathbf{ker}B$: these are B-vertical monos. Their cofibers (i.e. what's not hit by their image–their failure to be epi) are **isolated systems**, i.e. systems whose behaviour is completely not observable—hence why, despite the fact a mono doesn't hit them, they don't prevent such a map from being an iso of behaviours. Ideally, a isolated systems is mapped to $\varnothing$ by B.

[1] We note two things. First, to get a category of behaviours one would need to pair M with a posetal lifting thereof. Second, needs to work enrichedly for general Ms. Both things are noticed in [BKPV11].

2. $\mathbf{Sys}^{\twoheadrightarrow} \cap \mathbf{ker} B$: these are B-vertical epis. Their fibers (i.e. their failure to be mono) are **redundant systems**, i.e. systems whose observable behaviour could be accounted for by a much smaller system, since many parts of it are doing the same thing.

The opposite of isolated and redundant, in coalgebraic automata theory, are *reachable* and *observable* [Wiß22]. Indeed, imagine a state machine whose initial state is completely disconnected from the rest of the states. Then such a machine would be isolated since it can't have any observable behaviour. Likewise, imagine a machine whose states are all bisimilar to each other. Then such a machine would be redundant, since a single state would achieve the same behaviour.

Observe that B-vertical maps form a class of weak equivalences, since they clearly contain all isos and satisfy the 3-for-2 property:

$$\text{if two } f, g, fg \text{ are } B\text{-vertical, so is the third.} \tag{1.4}$$

Together with the B-vertical maps, we can also ask which maps are cartesian, i.e. are solution to a 'relative design problem' of finding a universal way to realize a system given some behaviours:

$$\begin{array}{ccc} f^*\mathsf{S} & \overset{f_{\mathsf{S}}}{\dashrightarrow} & \mathsf{S} \\ & & \\ A & \xrightarrow{f} & B\mathsf{S} \end{array} \tag{1.5}$$

Here $f^*\mathsf{S}$ is the universal solution to the problem of finding a system over S (hence the relativity) whose behaviour is that picked by f in those of S (because, in particular, $B(f^*\mathsf{S}) = A$).

In particular, when S is terminal and B finite-limits preserving (e.g. when it is corepresentable), a cartesian lift corresponds to the solution of an absolute design problem:

$$\begin{array}{ccc} !_A^*1 & \overset{!_A^*1}{\dashrightarrow} & 1 \\ & & \\ A & \xrightarrow{!_A} & 1 \end{array} \tag{1.6}$$

In other words, $!_A^*1$ is the **universal system with behaviour** A.

In automata theory, such a system is called the *minimal realization* [Moo56] of A, and it has received abundant categorical attention in the past (to mention some: [Gog72; AM74; BSW96; RSW98]).

So it is of interest to study when these cartesian lifts exists, i.e. to study the class of B-cartesian maps. If B were a fibration, we could conclude that all B-cartesian lifts exists and that, together with B-vertical maps, they form a cartesian factorization system [Mye21] on **Sys**.

This is, however, rarely the case, but not all hope is lost. We can still study when B-cartesian maps exists, and whether they form a factorization system—albeit with less nice properties than a cartesian one.

In particular we are interested in exploring when such a vertical–cartesian factorization system exists and when it is *reflective* [CHK85], i.e. when it induces a reflective subtheory of 'fully observable' systems.

Finally, this all interacts with the compositional structure of systems. For instance, when B is exact then we could form a 'quotient theory' $\mathbf{Sys}\left[\mathbf{ker}B^{-1}\right]$ corresponding to 'formally observable systems', inheriting the same compositions as $\mathbf{Sys}$. If B isn't exact then we have to restrict composition to those that preserve behaviour.

The issue of reflectivity and realization ties into this because we'd like to understand when the quotient $q : \mathbf{Sys} \to \mathbf{Sys}\left[\mathbf{ker}B^{-1}\right]$ admits a realization right adjoint.

Problem 1. *Describe* $\mathbf{Sys}\left[\mathbf{ker}B^{-1}\right]$ *and find sufficient conditions for its existence.*

Problem 2. *Define observable systems and observable processes.*

Another candidate object of 'formally observable systems' is the full image of q into $\mathbf{Set}$ (or the chosen behavioural theory). This is the theory whose interfaces, compositions and systems are the same as $\mathbf{Sys}$ but whose maps are those in the image (including the ones not hit by B–hence the *fullness*). One might recall from group theory that for an homomorphism $\varphi : G \to H$, $G/\mathbf{ker}\varphi \cong \operatorname{im}\varphi$. For categories we don't have this in general since the quotient by $\mathbf{ker}\varphi$ might not exist, but when it does B factors uniquely through it (by universal property of the quotient):

$$\begin{array}{ccc} \mathbf{Sys} & \xrightarrow{B} & \mathbf{Set} \\ {\scriptstyle \mathsf{bo}}\downarrow & \overset{\exists !}{\nearrow} & \uparrow{\scriptstyle \mathsf{ff}} \\ \mathbf{Sys}\left[\mathbf{ker}B^{-1}\right] & \xrightarrow[\mathsf{bo}]{} & \operatorname{im} B \end{array} \tag{1.7}$$

Meanwhile, the bijective-on-objects part of the $(\mathsf{bo}, \mathsf{ff})$-factorization of B factors though $\mathbf{Sys}\left[\mathbf{ker}B^{-1}\right]$, inducing the bijective-on-objects comparison map above bottom.

Problem 3. *When does* $\mathbf{Sys}\left[\mathbf{ker}B^{-1}\right] \cong \operatorname{im} B$*?*

2. Behaviour/realization adjunction

When B induces a vertical/cartesian (isolated/observable) factorization on $\mathbf{Sys}$, we get adjunctions between (tight) slices:

$$\mathbf{Sys}\left[\mathbf{ker}B^{-1}\right]/\mathsf{R} \underset{\hookrightarrow}{\overset{\text{obs}}{\underset{\perp}{\longleftarrow}}} \mathbf{Sys}/\mathsf{R} \tag{2.1}$$

The reflector is given by taking image:

$$\begin{array}{ccc} \mathsf{S} & \twoheadrightarrow & \operatorname{obs}\mathsf{S} \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle \operatorname{obs}\varphi} & \\ & \mathsf{R} & \end{array} \tag{2.2}$$

If $\mathbf{Sys}$ admits terminal systems, then this adjunctions descends to an adjunction between the theories themselves:

$$\mathbf{Sys}\left[\mathbf{ker}B^{-1}\right] \underset{\hookrightarrow}{\overset{\text{obs}}{\underset{\perp}{\longleftarrow}}} \mathbf{Sys} \tag{2.3}$$

3. Bisimilarity

Bisimilarity is a foundational idea in automata theory and the study of concurrent systems. It is a sufficient criterion to decree the behavioural equivalence of automata, by exhibiting a total and surjective relation between coalgebras.[2] Famously, it is not a necessary criterion in general: there exists *trace equivalent* coalgebras which are not bisimilar. However, bisimilarity makes proving trace equivalence easier.

We can recover this idea in our language, also clarifying where these notions and come from and how they relate to each other.

Definition 3.1. A bisimulation between systems S, T is a relation $\mathsf{R} : \mathsf{S} \nrightarrow \mathsf{T}$, meaning a span

$$\begin{array}{ccccc} & & \mathsf{R} & & \\ & \swarrow^{\pi_{\mathsf{S}}} & & \searrow^{\pi_{\mathsf{T}}} & \\ \mathsf{S} & & X & & \mathsf{T} \\ & \swarrow^{\pi_A} & & \searrow^{\pi_B} & \\ A & & & & B \end{array} \tag{3.1}$$

The following is a triviality:

Lemma 3.2. *If two systems are related by a bisimulation whose legs are B-vertical, then they are behaviourally equivalent.*

As promised, we recover the classical notions:

Example 3.1. From Example 1.1 and Example 1.2 we know all maps of automata over the same interface are behavioural equivalences, therefore a bisimulation of deterministic automata over the same interface is always witnessing a behavioural equivalence.

Example 3.2. From Example 1.3 we know maps of automata without an initial state are trace equivalences iff they are surjective on states, so that we recover bisimilarity of whole automata as a total and surjective relation.

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[2]Though bisimilarity is usually defined at the level of states, thus one speaks of bisimilar coalgebras when the relation is total and surjective, of when the coalgebra are equipped with initial states and they are bisimilar.

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