Logic for systems

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Abstract

Contents

1 Introduction

The study of systems is more often than not the study of things we want the systems to be, do, or *not* be or do. This might be because we design the systems for a purpose, or we want to understand what intrinsic purpose the system has, or to characterize its behaviour, or to generalize verify properties of systems. Broadly speaking, studying systems means *talking* about them and talking about things is what *logic* does.

In this work we introduce a basic toolset to talk about systems. We work within the framework of categorical systems theory, in which systems are organized in doubly indexed categories. As such, theories of systems are much like categories. And thus, much like for categories, we use fibrations to present logics.

This approach, which evolved from Lavwere's hyperdoctrines, is now well-established in categorical logic. The main reference is [**jacobs**]. In fact, another work of Jacobs, [**jacobs2**], can be considered a precursor of the present one. In it Jacobs considers fibrations of predicates over categories of coalgebras. We independently rediscovered this idea and some immediate developments theoref, such as the usage of comprehension to talk about invariant predicates.

In the following we introduce fibrations of theories of systems and unpack their logical meaning. Of particular interest are so-called behavioural predicates, which are fibrations of predicates obtained by pullback along a behaviour functor. This is, indeed, the main way in which logic is studied over systems, since behaviours are, in some sense, elements of systems [Mye22, §6]. The most common instance of this fact is to consider predicates on states, which are indeed a very simple kind of behaviour.

2 Preliminaries

Definition 2.1. A **theory of composition** is a symmetric monoidal double category $\mathbb{C}$. Its objects and tight 1-cells are **interfaces** and **maps** thereof, while its loose 1-cells are **composition operations** or **patterns**, and squares are **maps** thereof.

We often denote the monoidal product of $A, B \in \mathbb{C}$ as 'A, B'. In fact we'd rather consider $\mathbb{C}$ to be a coloured operad in **Cat**, i.e. a **double multicategory**. The reason we stick with symmetric monoidal categories is purely pedagogical. Thus we will often denote composition patterns $p : A_1, \ldots, A_n \to B$. This should be thought as an abstract composition operation for systems with interfaces A_1, A_2, $\ldots A_n$ that yields a system with interface B.

A theory of systems is then an algebra of such a multicategory, turning abstract into concrete composition:

Definition 2.2. A **theory of systems** over the composition theory $\mathbb{C}$ is a double functor

$$\mathbf{Sys} : \mathbb{C}^\top \xrightarrow{\text{unitary lax}} \mathbb{C}\mathbf{at}. \tag{2.1}$$

Concretely, **Sys** maps interfaces to **categories of systems**, composition operations to **composition functors**, maps of interfaces to **mapping profunctors** and maps of composition of operations to **transformations of composition functors**.

Hence given an interface $A \in \mathbb{C}$, we think of the objects of $\mathbf{Sys}(A)$ as systems of a certain kind while the maps are **simulations** between them, i.e. some notion of structure-preserving map between them.

$$\mathbf{Sys}(A) = \left\{ \ \mathsf{S} \xrightarrow{\varphi} \mathsf{T} \ \right\} \tag{2.2}$$

Each composition $p : A_1, \ldots, A_n \to B$ is substantiated as a functor

$$- \cdot p : \mathbf{Sys}(A_1) \times \cdots \times \mathbf{Sys}(A_n) \to \mathbf{Sys}(B) \tag{2.3}$$

and this assignment is strictly (multi)functorial, meaning $(\mathsf{S} \cdot p) \cdot q = \mathsf{S} \cdot pq$.

Meanwhile, map of interfaces induce profunctors which give notions of simulations between systems on different interfaces:

$$\mathbf{Sys}(A \xrightarrow{k} A') : \mathbf{Sys}(A) \nrightarrow \mathbf{Sys}(A') \tag{2.4}$$

Hence an element $\ell \in \mathbf{Sys}(A \xrightarrow{k} A')(\mathsf{S}, \mathsf{T})$ is a **simulation of** S **in** T **mediated by the maps of interfaces** k. This assignment is correspondingly unitary lax functorial, meaning that $\mathbf{Sys}(A = A)$ is the hom-profunctor of $\mathbf{Sys}(A)$, and that for composable maps of interfaces $A \xrightarrow{k} A' \xrightarrow{\ell} A''$ there are coherent natural transformations $\mathbf{Sys}(k) \odot \mathbf{Sys}(\ell) \Rightarrow \mathbf{Sys}(k \mathbin{⨟} \ell)$ which compose morphisms of systems.[1]

Finally, squares in $\mathbb{C}$ induce squares witnessing the extension of a simulation of systems along a map of composition of operations:

$$\begin{array}{ccc} \mathbf{Sys}(A_1) \times \cdots \times \mathbf{Sys}(A_n) & \xrightarrow{\mathbf{Sys}(k_1) \times \cdots \times \mathbf{Sys}(k_n)} & \mathbf{Sys}(A_1') \times \cdots \times \mathbf{Sys}(A_n') \\ {\scriptstyle - \cdot p} \downarrow & \Downarrow {\scriptstyle \mathbf{Sys}(\alpha)} & \downarrow {\scriptstyle - \cdot p'} \\ \mathbf{Sys}(B) & \xrightarrow[\mathbf{Sys}(h)]{} & \mathbf{Sys}(B') \end{array} \tag{2.5}$$

We can thus consider $\mathbf{Sys} : \mathbb{A}\mathbf{lg}(\mathbb{C})$ a kind of category, by leaving the indexing on interfaces implicit. Its objects $\mathsf{S} \in \mathbf{Sys}$ are systems $\mathsf{S} \in \mathbf{Sys}(A)$ over a certain (left implicit) interface $A \in \mathbb{C}$, and whose hom-sets $\mathbf{Sys}(\mathsf{S}, \mathsf{T})$ are given by maps of systems $\varphi \in \mathbf{Sys}(h)(\mathsf{S}, \mathsf{T})$, as h varies among maps of interfaces.

In fact, this defines a sort of Grothendieck construction of $\mathbf{Sys}$, in the form of a double category whose objects are pairs $(A \in \mathbb{C}, \mathsf{S} \in \mathbf{Sys}(A))$, denoted as $\binom{\mathsf{S}}{A}$, whose tight 1-cells are maps of systems and interfaces, and whose loose 1-cells $\binom{\mathsf{S}}{A} \to \binom{\mathsf{T}}{B}$ are compositions $p : A \to B$ such that $\mathsf{S} \cdot p = \mathsf{T}$.

We denote this double category as $\mathbb{S}\mathbf{ys}$. We make good use of the tight part of this double category. In there, we can define universal constructions of systems.

Definition 2.3. A **diagram of systems** is a category $\mathcal{T}$ together with a functor $\mathsf{T} : \mathcal{T} \to \mathbb{S}\mathbf{ys}$ picking out a system T_t with interface I_t for each $t \in \mathcal{T}$ and a map of systems $\mathsf{T}_f : \mathsf{T}_t \to \mathsf{T}_s$ over the map of interfaces $I_t \to I_s$ for each 1-cell $f : t \to s \in \mathcal{T}$.

Definition 2.4. The **limit** (resp. **colimit**) of a diagram of systems is the categorical limit (resp. colimit) of the diagram in the tight category of $\mathbb{S}\mathbf{ys}$.

Definition 2.5. A **lax map of systems theories** is a doubly laxly indexed functor:

$$\begin{array}{ccc} \mathbb{C}^\top & \xrightarrow{\mathbf{Sys}} & \\ {\scriptstyle F^\top} \downarrow & \Downarrow {\scriptstyle \bar{F}} & \mathbb{C}\mathbf{at} \\ \mathbb{C}'^\top & \xrightarrow[\mathbf{Sys}']{} & \end{array} \tag{2.6}$$

[1] $\odot$ denotes composition of profunctors.

This means that $F : \mathbb{C} \to \mathbb{C}'$ is a lax double functor, while $\bar{F}$ is a lax natural, monoidal family of functors

$$F_A : \mathbf{Sys}(A) \longrightarrow \mathbf{Sys}(F(A)), \quad A \in \mathbb{C}. \tag{2.7}$$

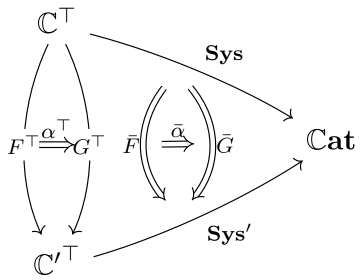
Specifically, this means that for each composition pattern $p : A_1, \ldots, A_n \to B$ there is a functorial choice of squares

$$\begin{array}{ccc} \mathbf{Sys}(A_1) \times \cdots \times \mathbf{Sys}(A_n) & \xrightarrow{\;-\cdot p\;} & \mathbf{Sys}(B) \\ {\scriptstyle F_{A_1} \times \cdots \times F_{A_n}} \big\downarrow & \overset{\ell_p}{\nearrow\!\!\!\nearrow} & \big\downarrow {\scriptstyle F_B} \\ \mathbf{Sys}(F(A_1)) \times \cdots \times \mathbf{Sys}(F(A_n)) & \xrightarrow[-\cdot F(p)]{} & \mathbf{Sys}(F(B)) \end{array} \tag{2.8}$$

A **map of system theories** is **taut** when F is a strong double functor and $\bar{F}$ is pseudonatural.

Remark 2.5.1. A category like $\mathcal{T}$ can be turned into a double category by equipping it with trivial loose moprhisms only, and then into a trivial theory of systems by assigning to each $t \in \mathcal{T}$ the terminal category 1. Then a diagram $\mathsf{T} : \mathcal{T} \to \mathbf{Sys}$ is a (necessarily taut) map of theories.

Definition 2.6. A **map of lax maps of systems theories** is a natural transformation



(2.9)

This amounts to a tight natural transformation $\alpha : F \Rightarrow G$, as well as a lax natural family of natural transformations $\bar{\alpha}_A : \bar{F}_A \Rightarrow \bar{G}_A$ for each $A \in \mathbb{C}$.

Thus theories, lax maps and maps thereof form a 2-category $\mathbb{S}\mathbf{ys}\mathbb{T}\mathbf{h}$.

2.1 Behaviour

A behaviour functor is a way to observe systems. Observations are usually taken to be sets, or more generally, objects of a topos. In practice any reasonably structured category can be used as a target for behaviour, and one might even consider any functor out of a theory of systems to be a behaviour.

Definition 2.7. The **behavioural theory** associated to a finitely complete category $\mathcal{E}$ is a theory of systems over the cartesian double category $\mathbb{S}\mathbf{pan}(\mathcal{E})$. It is defined as follows:

1. An interface $A \in \mathbb{S}\mathbf{pan}(\mathcal{E})$ is associated to the slice category:

$$\mathcal{E}/A \tag{2.10}$$

   and $\mathcal{E}/A_1 \times \mathcal{E}/A_2 \to \mathcal{E}/A_1 \times A_2$ is given by cartesian product of morphisms.

2. A composition $A \xleftarrow{p} X \xrightarrow{q} B$ is sent to its pull-push functor:
$$\mathcal{E}/A \xrightarrow{p^*} \mathcal{E}/X \xrightarrow{q_*} \mathcal{E}/B. \tag{2.11}$$

3. A map $h : A \to A'$ is sent to the profunctor
$$\mathcal{E}/h : \mathcal{E}/A \nrightarrow \mathcal{E}/A' \tag{2.12}$$
defined as
$$\mathcal{E}/h(E \xrightarrow{\pi} A, E' \xrightarrow{\pi'} A') = \left\{ E \xrightarrow{\varphi} E' \;\middle|\; \begin{array}{ccc} E & \xrightarrow{\varphi} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ A & \xrightarrow[h]{} & A' \end{array} \right\} \tag{2.13}$$
The compositors are then defined by pasting squares.

4. A square of spans is sent to the natural transformation defined by pull-push of whole squares.

We denote this theory as $\mathcal{E} : \mathbb{A}\mathbf{lg}(\mathbb{S}\mathbf{pan}(\mathcal{E}))$.

Let $\mathbf{Sys} : \mathbb{A}\mathbf{lg}(\mathbb{C})$.

Definition 2.8. A **behaviour functor** for $\mathbf{Sys}$ is a lax map $\mathcal{B} : \mathbf{Sys} \to \mathcal{E}$ into some behavioural theory $\mathcal{E}$.

Proposition 2.9. *For each system* $\mathsf{T} \in \mathbf{Sys}$ *with interface* $I \in \mathbb{C}$*, there is a* ***corepresentable behaviour*** $\mathbf{Sys}(\mathsf{T}, -) : \mathbf{Sys} \to \mathbf{Set}$ *given on each* $A \in \mathbb{C}$ *by the functor*
$$\mathbf{Sys}(\mathsf{T}, -) : \mathbf{Sys}(A) \longrightarrow \mathbf{Set}/\mathbb{C}(I, A) \tag{2.14}$$
defined as $\mathbf{Sys}(\mathsf{T}, \mathsf{X}) = \sum_{h \in \mathbb{C}(I,A)} \mathbf{Sys}(h)(\mathsf{T}, \mathsf{X}) \xrightarrow{\mathsf{fst}} \mathbb{C}(I, A)$.

Proposition 2.10. *For each diagram of systems* $\mathsf{T} : \mathcal{T} \to \mathbf{Sys}$*, there is a* ***jointly corepresentable behaviour*** $\mathbf{Sys}(\mathsf{T}, -)$ *given on each* $A \in \mathbb{C}$ *by the functor*
$$\mathbf{Sys}(\mathsf{T}, -) : \mathbf{Sys}(A) \longrightarrow \mathbf{Set}/\operatorname{colim}_{t \in \mathcal{T}} \mathbb{C}(I_t, A) \tag{2.15}$$
defined as $\mathbf{Sys}(\mathsf{T}, \mathsf{X}) = \operatorname{colim}_{t \in \mathcal{T}} \left(\sum_{h \in \mathbb{C}(I_t,A)} \mathbf{Sys}(h)(\mathsf{T}_t, \mathsf{X}) \xrightarrow{\mathsf{fst}} \mathbb{C}(I_t, A) \right)$.

Proposition 2.11. *Every jointly corepresentable behaviour factors as*
$$\begin{array}{ccc} \mathbf{Sys} & \xrightarrow{\mathbf{Sys}(\mathsf{T},-)} & \mathbf{Set} \\ & {\scriptstyle \mathbf{Sys}(\mathsf{T}_{(-)},-)} \searrow \quad \nearrow {\scriptstyle \operatorname{colim}_{t \in \mathcal{T}}} & \\ & \mathbf{Psh}(\mathcal{T}) & \end{array} \tag{2.16}$$

Proof. Explicitly, the map $\mathbf{Sys}(\mathsf{T}_{(-)}, -)$ is defined on a system X as
$$\begin{aligned} \mathbf{Sys}(\mathsf{T}_{(-)}, \mathsf{X}) : \mathcal{T} &\longrightarrow \mathbf{Set} \\ t &\mapsto \mathbf{Sys}(\mathsf{T}_t, \mathsf{X}), \end{aligned} \tag{2.17}$$
making the claimed commutativity evident. □

Proposition 2.12. *Suppose* $\mathsf{T} : \mathcal{T} \to \mathbf{Sys}$ *is cocontinuous, meaning* $\mathsf{T}_{\operatorname{colim}_i t_i} \cong \operatorname{colim}_i \mathsf{T}_{t_i}$*, then* $\mathbf{Sys}(\mathsf{T}_{(-)}, -)$ *factors through the inclusion* $\mathbf{Sh}(\mathcal{T}) \hookrightarrow \mathbf{Psh}(\mathcal{T})$.

3 Fibrations of goals

Definition 3.1. A **fibration of goals** is a fibration of systems theories

$$\mathbb{C}^{\top} \underset{\mathbf{Sys}}{\overset{\mathbf{Goals}}{\underset{\Downarrow \pi}{\rightrightarrows}}} \mathbb{C}\mathbf{at} \tag{3.1}$$

such that π is strictly natural.

Concretely, this amounts to

1. a family of fibrations

$$\begin{array}{c} \mathbf{Goals}(I) \\ \downarrow{\scriptstyle u_I} \\ \mathbf{Sys}(I) \end{array} \tag{3.2}$$

indexed by interfaces $I : \mathbb{C}$. This means that for each system $\mathsf{S} : \mathbf{Sys}(I)$ there is a category $u_I^{-1}(\mathsf{S})$ of goals for that system, and that any map of systems $\varphi : \mathsf{S} \to \pi\binom{G}{\mathsf{R}}$ lifts to a map of goals $\varphi^*\binom{G}{\mathsf{R}} \to \binom{G}{\mathsf{R}}$. One thinks of the goal $\varphi^*\binom{G}{\mathsf{R}}$ over S as the goal S inherits once it is simulated by R. The cartesian map $\varphi^* G \to G$ substantiates this interpretation, but most importantly, the examples do.

2. For each process $p : I \multimap K$ in $\mathbb{C}$, a commutative square of functors:

$$\begin{array}{ccc} \mathbf{Goals}(I) & \xrightarrow{u_I} & \mathbf{Sys}(I) \\ {\scriptstyle \mathbf{Goals}(p)}\downarrow & & \downarrow{\scriptstyle \mathbf{Sys}(p)} \\ \mathbf{Goals}(K) & \xrightarrow[u_K]{} & \mathbf{Sys}(K) \end{array} \tag{3.3}$$

This means that a goal G on a system $\mathsf{S} : \mathbf{Sys}(I)$ can be covariantly pushed through a process $p : I \to K$ to get new goal $\mathbf{Goals}(p)(G)$ for the reindexed system $\mathbf{Sys}(p)(\mathsf{S}) : \mathbf{Sys}(K)$.

3. For each map of interfaces $h : I \longrightarrow J$, a square in $\mathbb{C}\mathbf{at}$:

$$\begin{array}{ccc} \mathbf{Goals}(I) & \xrightarrow{u_I} & \mathbf{Sys}(I) \\ {\scriptstyle \mathbf{Goals}(h)}\downarrow & \overset{}{\underset{u_h}{\Longrightarrow}} & \downarrow{\scriptstyle \mathbf{Sys}(h)} \\ \mathbf{Goals}(J) & \xrightarrow[u_J]{} & \mathbf{Sys}(J) \end{array} \tag{3.4}$$

Moreover, they are required to satisfy the conditions one can find in [Gra19, Definition 3.2.7(b)], namely:

1. **Horizontal naturality**. For every consecutive pair of maps of interfaces $I \xrightarrow{h} J \xrightarrow{k} H$, we have:

Matteo: todo

2. **Square naturality**. For every map of processes $\begin{matrix} I & \xrightarrow{h} & J \\ p\downarrow & \xRightarrow{\alpha} & \downarrow q \\ K & \xrightarrow[k]{} & G \end{matrix}$, we have:

Matteo: todo

Example 3.1. Let $\mathbf{Moore}_{(F,T)}$ (from now on just **Moore**), where $F : \mathcal{E} \to \mathcal{S}$ is a fibration and T a section thereof, be a theory of Moore machines. A very simple notion of goal we can define for these systems is that of a subobject of preferred states $G \subseteq S$. This means $\mathbf{Goals}\binom{I}{O}$ is the category $\mathsf{states}^{\subseteq}\binom{I}{O}$ of Moore machines equipped with a subobject of their state space, and whose morphisms are goal-preserving maps of Moore machines:[2]

$$\begin{array}{ccc} & G \subseteq H & \\ \swarrow & & \searrow \\ S & \xrightarrow{\varphi} & R \\ \binom{TS}{S} & \overset{T\varphi}{\underset{\varphi}{\rightrightarrows}} & \binom{TR}{R} \\ f\downarrow\uparrow f^\sharp & & g\downarrow\uparrow g^\sharp \\ \binom{I}{O} & = & \binom{I}{O} \end{array} \tag{3.5}$$

$\mathsf{states}^{\subseteq}\binom{I}{O}$ is evidently fibred over $\mathbf{Moore}\binom{I}{O}$ by forgetting the subspace.

Notice we could also choose a dual notion of morphism of goals: instead of asking φ to preserve preferred states, we could ask for it to reflect them, that is, to make sure only states from G end up in H, even though not all states in G might be sent there. This amounts to pullback H along φ and ask that $\varphi^* H \subseteq G$.

This is always the case: given a fibration of goals and goal-preserving maps, its dual fibration [**benabou**] is a fibration of goals and goal-reflecting maps, and *vice versa*.

Let's define the rest of the structure of π, just to acquaint ourselves with their form. First of all, we need to say how $\mathsf{states}^{\subseteq}$ acts on lenses, charts and squares. Since rewiring a Moore machine along a lens $\binom{p^\sharp}{p} : \binom{I}{O} \rightrightarrows \binom{J}{Q}$ doesn't change its state space, the action of $\mathsf{states}^{\subseteq}\binom{p^\sharp}{p}$ on the vertical arrows is the identity. Similarly, given a chart $\binom{h^\flat}{h} : \binom{I}{O} \rightrightarrows \binom{J}{Q}$, the profunctor $\mathsf{states}^{\subseteq}\binom{h^\flat}{h}$

$$\begin{aligned} \mathsf{states}^{\subseteq}\binom{h^\flat}{h} : \mathsf{states}^{\subseteq}\binom{I}{O}^{\mathsf{op}} \times \mathsf{states}^{\subseteq}\binom{J}{Q} &\longrightarrow \mathbf{Set} \\ \binom{S^*}{\mathsf{S}}, \binom{R^*}{\mathsf{R}} &\longmapsto \left\{\varphi \in \mathbf{Moore}\binom{h^\flat}{h}(\mathsf{S}, \mathsf{R}) \mid \varphi(S^*) \subseteq R^*\right\} \end{aligned} \tag{3.6}$$

naturality squares ((3.3) and (3.4)) for $\mathsf{states}^{\subseteq}$.

Example 3.2. The goals in the last example ignore most of the data of a Moore machine. One can come up with different notions of goal that take more data into consideration. For instance,

[2]Of course we need at least to be able to pullback subobjects in the ambient category.

assuming $\mathcal{C}$ is regular, one could also specify preferred input/outputs:

$$u^{-1}_{\binom{I}{O}}(S, \mathsf{expose}, \mathsf{update}) := \left\{ \begin{array}{ccc} U & \subseteq & V \\ & \searrow \quad \swarrow & \\ & S \times I \times O & \end{array} \right\} \tag{3.7}$$

Regularity allows us to push-pull such a goal along the span $I \times O \xleftarrow{\langle f^\sharp, O\rangle} O \times J \xrightarrow{J \times f} J \times Q$ induced by a lens $\binom{f^\sharp}{f} : \binom{I}{O} \leftrightarrows \binom{J}{Q}$.

Example 3.3. Other kinds of goals on a Moore machine $\mathsf{S} = (S, \mathsf{expose}, \mathsf{update})$ are value functionals $c : S \to \mathbb{R}$ (assuming $\mathbb{R}$ is available) and order structures $(S, \leq)$. Both pull back by precomposition with maps of machines. Again, in both examples S could be replaced with less forgetful objects built from the entirety of S.

Example 3.4. On deterministic Moore machines **Moore**(**Set**), selection functions form a goal. Fix a set R, then for a machine S, the fiber of goals is given by selections $\mathsf{sel}(S) : (S \to R) \to PS$, with maps $\mathsf{sel}(S) \to \mathsf{sel}(T)$ given by maps $S \to T$ that make the obvious diagram commute. Again, this example can easily be generalized in multiple directions.

Example 3.5. The theory of Mealy machines $\mathbf{Mealy} : \mathbb{L}\mathbf{ens}(\mathcal{S})^{\mathrm{lop}\top} \xrightarrow{\text{unitary lax}} \mathbf{Cat}$ is defined on the process theory of lenses and charts in a cartesian category[3] $\mathcal{S}$, like the theory of Moore machines, but Mealy machines are reindexed by lenses contravariantly instead. The category $\mathbf{Mealy}\binom{I}{O}$ has objects Mealy machines in $\mathcal{S}$, hence pairs $(S : \mathcal{S}, \delta : S \times O \to S \times I)$ (notice the inversion between Outputs and Inputs), and morphisms being maps between state spaces that commute with the dynamics. Maps between Mealy machines over different interfaces are defined similarly, thus defining the profunctors that **Mealy** associates to charts.

Given a lens $\binom{p^\sharp}{p} : \binom{J}{Q} \leftrightarrows \binom{I}{O}$, it acts on $\mathbf{Mealy}\binom{I}{O}$ by sending a machine (S, δ) to:

$$\binom{p^\sharp}{p} \cdot (S, \delta) := \left(S : \mathcal{S},\ S \times Q \xrightarrow{S \times \langle p, Q\rangle} S \times O \times Q \xrightarrow{\delta \times Q} S \times I \times Q \xrightarrow{S \times p^\sharp} S \times J \right), \tag{3.8}$$

an operation which is clearly functorial.

Note that POMDPs [?] are a variant of **Mealy** machines where the dynamics δ is a Kleisli map for a probability monad, and thus are reindexed by optics (with options on the amount of generality) instead of plain lenses. We work in a simpler setting to bring home the point we want to make without too much noise, but it should be understood that generality is not an issue here.

Inspired by the observation of Hedges and Rodriguez Sakamoto [HS23] that 'value functions' reindex contravariantly along lenses, we can build a fibration of goals over **Mealy**(**Set**) (again, it's easy to generalize beyond this if desired). The goals over a Mealy machine $(X, \delta) \in \mathbf{Mealy}\binom{A}{O}$, thus with observations O, actions A and dynamics $\delta : X \times O \to X \times A$ are defined to be 'utility functions' $U : X \times O \to \mathbb{R}$. They are contravariantly reindexed along lenses like Mealy machines:

[3]This definition can be easily extended to the monoidal setting by employing optics [**optics**], as well as to the fibred setting by using generalized lenses [Spi19].

given a utility U over (S, δ), and a lens $\binom{p^\sharp}{p} : \binom{A'}{O'} \leftrightarrows \binom{A}{O}$, we get

$$\binom{p^\sharp}{p} \cdot U : X \times O' \overset{X \times p}{\to} X \times O \overset{U}{\to} \mathbb{R}. \tag{3.9}$$

Note that POMDPs are usually defined as equipped with a reward function $r : X \times A \to \mathbb{R}$, but these have bad compositional properties—by using functions $X \times O \to \mathbb{R}$ we strictly generalize the latter (since every such r induces a utility $\delta \fatsemi r$) and make them compose contravariantly along lenses just like Mealy machines.[4]

3.1 Behavioural goals

By far, the most common goals on a system have to do with its behaviour. For instance one might ask for a dynamical system to be in a stable equilibrium, or assign an action functional to its trajectories. All these instances are obtained by by pulling back a fibration of predicates along a behaviour.

Recall a **behaviour** for a system theory **Sys** is a morphism of system theories $B : \mathbf{Sys} \to \mathbf{Set}$, where **Set** is the observational theory on **Set**. Hence a fibration $\pi : \mathbf{Goals} \to \mathbf{Set}$ induces a fibration on **Sys** by pullback:[5]

$$\begin{array}{ccc} \binom{G}{\mathsf{S}} & \longmapsto & \binom{G}{B^\flat(\mathsf{S})} \\ \big\downarrow & \begin{array}{ccc} B^*\mathbf{Pred} & \longrightarrow & \mathbf{Pred} \\ {\scriptstyle B^*\pi}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ \mathbf{Sys} & \xrightarrow{B} & \mathbf{Set} \end{array} & \big\downarrow \\ \mathsf{S} & \longmapsto & B^\flat(\mathsf{S}) \end{array} \tag{3.10}$$

In $B^*\mathbf{Pred}$ is indexed by the process theory $\mathbb{C}$ of **Sys**. For a fixed interface $I : \mathbb{C}$, objects are pairs of a system $\mathsf{S} : \mathbf{Sys}(I)$ and a predicate $G : \mathbf{Pred}(B(I))$ such that $\pi(G) = B^\flat(\mathsf{S})$. Morphisms are maps of systems $\varphi : \mathsf{S} \to \mathsf{R} : \mathbf{Sys}(I)$ together with maps of predicates $f : G \to B^\flat(\varphi)^* H$ in $\mathbf{Goals}(B(I))$.

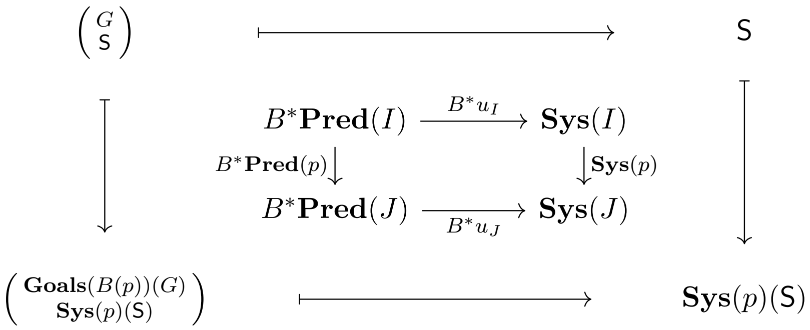
Reindexing along a process $p : I \mathrel{-\!\!\bullet\!\!\to} J$ is given by

$$B^*\mathbf{Pred}(p)\binom{G}{\mathsf{S}} = \binom{\mathbf{Goals}(B(p))(G)}{\mathbf{Sys}(p)(\mathsf{S})} \tag{3.11}$$

[4] For a covariant theory, Moore machines are the right choice.

[5] In particular, a strict 2-pullback in $\mathbb{S}\mathbf{ys}\mathbb{T}\mathbf{h}$.

which makes $B^*\pi$ natural, since the following evidently commutes



(3.12)

Moreover, since fibrations are pullback-stable, $B^*\pi$ is still a fibration.

Example 3.6. Preferred substates in an open dynamical system (Example 3.1) are behavioural predicates (up to replacing **Set** with $\mathbf{Set}(\mathcal{C})$). They are given by pulling back the subobject fibration on **Set** along $\mathsf{states} : \mathbf{Moore} \to \mathbf{Set}(\mathcal{C})$, the behaviour sending a Moore machine to its state space (while processes and maps thereof are all forgotten, i.e. sent to the terminal object in $\mathcal{C}$). Similarly, Example 3.2 is also an example of behavioural predicate.

Example 3.7. Let $\mathsf{trajs} : \mathbf{Moore} \to \mathbf{Set}$ be the theory of behaviour of trajectories of Moore machines. Then $\mathsf{trajs}^{\subseteq} := \mathsf{trajs}^*\mathbf{Set}^{\subseteq}$ is a theory of predicates given by predicates on trajectories of Moore machines.

There are various adjoint functors going back and forth between predicates over states and trajectories:

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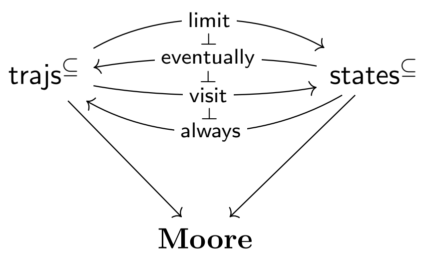


(3.13)

The predicate $\mathsf{eventually}(\varphi)$ is the predicate on trajectories given by

$$\mathsf{eventually}(\varphi) := \{(t_n)_{n\in\mathbb{N}} \mid \exists n_0 \in \mathbb{N}\ \forall n \geq n_0, \varphi(t_n)\}. \tag{3.14}$$

If the fibers of $\mathsf{states}^{\subseteq}$ *are finite*,[6] then $\mathsf{eventually}$ preserves conjunctions and disjunctions, hence admits both adjoints. The left one sends a set of trajectories to the largest set of states they all eventually end up in:

$$\mathsf{limit}(\psi) = \bigcap_{\varphi\subseteq S} \{\forall t \in \mathsf{trajs}(S), \psi(t) \implies \mathsf{eventually}(\varphi)(t)\} \tag{3.15}$$

[6] Otherwise one can have infinite conjunctions of predicates which are satisfied by an unbounded sequence of n_0s.

Notice this condition cannot be stated 'pointwise', i.e. this is not the set of limit points of the trajectories in ψ—for instance, trajectories in ψ might all be periodic in which case $\mathsf{limit}(\psi)$ is the set of orbits they trace and thus no state is a limit point *per se*.

We have

$$\mathsf{eventually}(\varphi) \leq \psi \iff \varphi \leq \mathsf{limit}(\psi) \tag{3.16}$$

In fact if every trajectory which eventually satisfies φ satisfies ψ then φ implies...

The right adjoint to $\mathsf{eventually}$ sends a set of trajectories to the smallest set of states they all visit:

$$\mathsf{visit}(\psi) = \bigcup_{\varphi \subseteq S} \{\forall t \in \mathsf{eventually}(\varphi),\ t \in \psi\} \tag{3.17}$$

Furthemore, since this latter functor preserves disjunctions, it admits an ulterior right adjoint mapping a set of states to the set of trajectories which never leave it.

Having all these adjoints induces modalities on each of $\mathsf{trajs}^{\subseteq}$ and $\mathsf{states}^{\subseteq}$. For instance, the composite $\lhd = \mathsf{eventually} \fatsemi \mathsf{limit}$ is a comonad which we can call the 'attractor comonad', meaning $\lhd\varphi$ is eventually satisfied when starting in a state satisfying φ, and that $\lhd\varphi$ is minimal as such.

4 Using fibrations of predicates

Expressing goals as 'fibrations of predicates' means we can now study goal-seeking behaviour with the tools of categorical logic.

4.1 Realizing goals with comprehension

Let's consider, along with a fibration of goals $\pi : \mathbf{Goals} \to \mathbf{Sys}$, the following string of adjoints:

$$\mathbf{Goals} \quad 0 \dashv \pi \dashv 1 \dashv \{-\} \quad \mathbf{Sys} \tag{4.1}$$

Here both 0 and 1 are assumed to be sections of p, and they pick, respectively, an initial and terminal object to every fiber of p.

The meanings of these adjoints for a fibration of goals are:

1. A section $0 : \mathbf{Sys} \to \mathbf{Goals}$ left adjoint to π equips every system with an unsatisfiable goal. In fact, the adjunction says there is a natural isomorphism

$$\mathbf{Goals}\left(\binom{0_{\mathsf{S}}}{\mathsf{S}}, \binom{G}{\mathsf{R}}\right) \cong \mathbf{Sys}(\mathsf{S}, \mathsf{R}) \tag{4.2}$$

meaning that 0_{S} is preserved by any map of goals whatsoever.

2. A section $1 : \mathbf{Sys} \to \mathbf{Goals}$ right adjoint to π equips every system with a trivial(ly satisfiable) goal. In fact, the adjunction says there is a natural isomorphism

$$\mathbf{Goals}\left(\binom{G}{\mathsf{S}}, \binom{1_{\mathsf{R}}}{\mathsf{R}}\right) \cong \mathbf{Sys}(\mathsf{S}, \mathsf{R}) \tag{4.3}$$

meaning every goal is trivially preserved when mapping to 1_{S}.

3. A further right adjoint $\{-\} : \mathbf{Goals} \to \mathbf{Sys}$ realizes every goal as a system that satisfies it. In fact, the adjunction says there is a natural isomorphism

$$\mathbf{Goals}\left(\binom{1_{\mathsf{S}}}{\mathsf{S}}, \binom{G}{\mathsf{R}}\right) \cong \mathbf{Sys}\left(\mathsf{S}, \begin{Bmatrix} G \\ \mathsf{R} \end{Bmatrix}\right). \tag{4.4}$$

Since 1_{S} is the maximal goal, mapping from it means satisfying G, and if this is the same as mapping from S into $\left\{ {G \atop \mathsf{R}} \right\}$, it means every map of systems with goals that satisfies G can be given as a map into $\left\{ {G \atop \mathsf{R}} \right\}$, i.e. $\left\{ {G \atop \mathsf{R}} \right\}$ realizes G.

We observe the unit of $0 \dashv \pi$ and the counit of $\pi \dashv 1$ are trivial, essentially by assumption: we postulated 0 and 1 to be sections (right inverses) of π. A non-trivial consequence of this is the unit of $1 \dashv \{-\}$ is also trivial, since $1 \fatsemi \pi \dashv 1 \fatsemi \{-\}$.

The non-trivial units and counits of the adjunctions $0 \dashv \pi \dashv 1 \dashv \{-\}$ give us helpful comparison 2-cells:

1. Maps witnessing every goal gets a map from and to the trivial ones, given by the counit of $0 \dashv \pi$ and the unit of $\pi \dashv 1$, respectively:

$$\begin{aligned} ?_{\binom{G}{\mathsf{S}}} &\in \mathbf{Goals}\left(\binom{0}{\mathsf{S}}, \binom{G}{\mathsf{S}}\right) \\ !_{\binom{G}{\mathsf{S}}} &\in \mathbf{Goals}\left(\binom{G}{\mathsf{S}}, \binom{1}{\mathsf{S}}\right) \end{aligned} \tag{4.5}$$

2. A map witnessing the trivial goal on the realization of a goal G realizes G, given by the counit of $1 \dashv \{-\}$

$$\varepsilon_{\binom{G}{\mathsf{S}}} \in \mathbf{Goals}\left(\binom{1}{\left\{ {G \atop \mathsf{S}} \right\}}, \binom{G}{S}\right) \tag{4.6}$$

This map is not trivial in general since $\left\{ {G \atop \mathsf{S}} \right\}$ might be sizeably smaller than S.

A bit more indirectly we obtain a third important comparison map:

Lemma 4.1. *Let $\pi \dashv 1 \dashv \{-\} : \mathbf{Goals} \to \mathbf{Sys}$ be a realizable goal structure. Let $\binom{G}{\mathsf{S}} : \mathbf{Goals}$ be a system with goal. Then there are natural comparison maps*

$$\begin{aligned} \rho_{\binom{G}{\mathsf{S}}} &:= \begin{Bmatrix} G \\ \mathsf{S} \end{Bmatrix} \xrightarrow{\left\{ !_{\binom{G}{\mathsf{S}}} \right\}} \begin{Bmatrix} 1 \\ S \end{Bmatrix} = \mathsf{S} \\ \lambda_{\binom{G}{\mathsf{S}}} &:= \mathsf{S} = \begin{Bmatrix} 0 \\ \mathsf{S} \end{Bmatrix} \xrightarrow{\left\{ ?_{\binom{G}{\mathsf{S}}} \right\}} \begin{Bmatrix} G \\ \mathsf{S} \end{Bmatrix}. \end{aligned} \tag{4.7}$$

Example 4.1. Let's look again at Example 3.1. Can we find adjoints to $\pi : \mathsf{states}^{\subseteq}(\mathcal{C}, T) \to$ **Moore** as above?

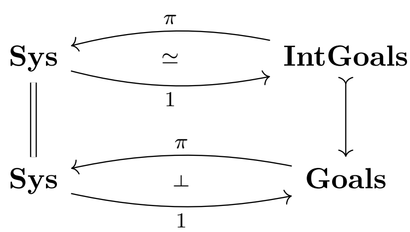
1. If $\mathcal{C}$ has an initial object, then we can define $0 : \mathbf{Moore} \to \mathsf{states}^{\subseteq}$ by equipping every Moore machine with the initial subobject $0 \subseteq S$ of its state space. This is evidently a section and since there is exactly one way to map from this subobject to any other, it forms a left adjoint to π.

2. Dually, we can define $1 : \mathbf{Moore} \to \mathsf{states}^{\subseteq}$ by equipping a given Moore machine with the terminal subobject $S = S$.

3. By letting ourselves be guided by the intuition we suggested above, let's build the comprehension structure. Given a subspace $S^* \subseteq S$ of a Moore machine $(S, \mathsf{expose}, \mathsf{update})$ with interface $\binom{I}{O}$, we want to find a Moore machine that always satisfies this goal, thus that always lives in S^*. The most obvious way to do this is to restrict the machine to the states S^*. But while we can restrict the expose map very easily, we can't corestrict the transition map as easily because there's no guarantee that $\mathsf{update}(s^*, i) \in S^*$ for $s^* \in S^*$ and $i \in I$! There's two possible solutions to this problem. One is to restrict the interface of the machine to only those inputs $I^* \subseteq I$ that always guarantee a valid transition. This is not viable though: morphisms of system theories (such as $\{-\}$) have to respect the indexing of interfaces. The second solution is to further restrict S^* to those states whose transitions all land in S^* again. In coalgebraic logic, this would be the subobject $\Box S^*$.

Definition 4.2. A **realizable goal structure** is a goal structure $\pi : \mathbf{Goals} \to \mathbf{Sys}$ which admits two further right adjoints $\pi \dashv 1 \dashv \{-\}$.

Intrinsic goals. It's interesting to look at goals for which the map ρ is invertible. These are given by looking at the fixpoints [PT91, §2-A] of the (fibred) adjunction $1 \dashv \{-\} : \mathbf{Goals} \to \mathbf{Sys}$. The fixpoint of an adjunction is the largest equivalence it co/restricts to. In our case, since 1 is already a right inverse, this means only restricting **Goals** to those objects for which the counit ρ described above is invertible, as desired.



(4.8)

We call the fixpoints of $1 \dashv \{-\}$ *intrinsic goals* since they capture those goals $\binom{G}{\mathsf{S}}$ which are already realized in the system. Hence S will appear *as if* it is pursuing G.

Definition 4.3. An **intrinsic goal** G of the system S is one that meets either of the following equivalent definitions:

1. $\rho : \left\{ {G \atop \mathsf{S}} \right\} \to \mathsf{S}$ is invertible,

2. the counit $G \to 1_{\mathsf{S}}$ is invertible.

Clearly trivial goals are always intrinsic, but in some cases there might be more.

Lemma 4.4. *Let* **Goals** *be a theory of goals with trivial goals and whose fibers are preorders. Let* $\gamma : G \to H : \mathbf{Goals}_{\mathsf{S}}$ *be a map of goals over* $\mathsf{S} : \mathbf{Sys}$*. Then if* G *is intrisic, so is* H*.*

Proof. If G is intrisic, $1_{\mathsf{S}} \leq G \leq H$, making H intrisic. □

In other situations, a morphism $1_{\mathsf{S}} \to H$ is not enough to conclude that $1_{\mathsf{S}} \cong H$, because in general there can be non-trivial morphisms out of a terminal object.

4.2 Adjoints to reindexing

A fibration has existential quantifiers iff all reindexing functors have left adjoints. So what does a left adjoint to reindexing mean for a fibration of goals?

Let $\varphi : \mathsf{S} \to \mathsf{R}$ be a map of systems over the same interface I, then consider the adjunction:

$$\mathbf{Goals}(I)(\mathsf{S}) \underset{\varphi^*}{\overset{\Sigma_\varphi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Goals}(I)(\mathsf{R}) \tag{4.9}$$

This would mean

$$\mathbf{Goals}(I)(\mathsf{R})(\Sigma_\varphi G, H) \cong \mathbf{Goals}(I)(\mathsf{S})(G, \varphi^* H) \tag{4.10}$$

Matteo: ...

4.3 Behavioural vs natural goals

There is a 'natural' fibration of predicates on a given category, where predicates are identified with subobjects. The same happens for system theories: for suitably complete **Sys**, there is a fibration of subsystems $\mathrm{cod} : \mathbf{Sys}^{\subseteq} \to \mathbf{Sys}$.

Suppose we pulled back $\pi : \mathbf{Goals} \to \mathbf{Set}$ along $B : \mathbf{Sys} \to \mathbf{Set}$ to get a behavioural goal $B^*\pi : B^*\mathbf{Pred} \to \mathbf{Sys}$. When B preserves monos,[7] the latter fibration admits a comparison map from the fibration of subsystems, given by taking behaviours of the subsystems:

$$\begin{array}{ccc} \mathbf{Sys}^{\subseteq} & \xrightarrow{\quad B \quad} & B^*\mathbf{Pred} \\ & {\scriptstyle \mathrm{cod}} \searrow \quad \swarrow {\scriptstyle B^*\pi} & \\ & \mathbf{Sys} & \end{array} \tag{4.11}$$

When they exist, the adjoints to such a map are very interesting:

[7] Notably, corepresentable behaviours preserve monos [Rie17, Exercise 2.1(ii)].

$$\begin{tikzcd} \mathbf{Sys}^{\subseteq} \arrow[rr, "B" description] \arrow[dr, "\mathrm{cod}"'] & & B^*\mathbf{Pred} \arrow[ll, bend right, "\Diamond"'] \arrow[ll, bend left, "\square"] \arrow[dl, "B^*\pi"] \\ & \mathbf{Sys} & \end{tikzcd} \tag{4.12}$$

They take a behavioural goal G on a system S and turn it into a subsystem of S. The right adjoint $\square$ produces the largest subsystem that always exhibits G, while the left adjoint produces the smallest subsystem that can exhibit G.

Example 4.2. Let $\mathbf{Sys} = \mathbf{Coalg}$ and B be the behaviour of states (**??**). For a fixed T and a fixed T-coalgebra (S, δ), goals in $\mathsf{states}^{\subseteq}$ are predicates on the state space of the coalgebra which are, however, not compatible with the dynamics δ in any way. Hence after a step, a given goal $G \subseteq S$ might be completely obliterated.

On the other hand, goals in $\mathbf{Coalg}(T)^{\subseteq}$ over (S, δ) are subcoalgebras, hence a subobject $G \subseteq S$ *plus* a coalgebra structure $\delta|_G : G \to TG$ that witnesses the stability of G under the dynamics δ.

In this setting, the adjoint string (4.12) exists whenever $\mathcal{C}$ is regular and mono-co/cocomplete, i.e. whenever the posets of subobjects are co/complete. The two adjoints to B encode the 'possibly' and 'necessarily' operators, respectively.

These adjoints compose with the fundamental adjunction $\mathrm{cod} \dashv \mathrm{id} \dashv \mathrm{dom}$ to yield the rightmost two adjunctions in (4.1):

$$\begin{tikzcd} \mathbf{Sys} \arrow[r, "\mathrm{id}" description] & \mathbf{Sys}^{\subseteq} \arrow[l, bend right, "\mathrm{cod}"'] \arrow[l, bend left, "\mathrm{dom}"] \arrow[r, "B" description] & B^*\mathbf{Set}^{\subseteq} \arrow[l, bend right, "\Diamond"'] \arrow[l, bend left, "\square"] \end{tikzcd} \tag{4.13}$$

This should make clearer why $\{-\}$ in Example 4.1 behaves so much like $\square$. The reason $\{-\}$ looks like $\square$ and not $\Diamond$ is that the first, by virtue of being a right adjoint, preserves pullbacks and therefore exhibits a comprehension structure for $B^*\pi$.

Example 4.3. Often $B : \mathbf{Sys}^{\subseteq} \to B^*\mathbf{Set}^{\subseteq}$ is fully faithful hence exhibiting forward-invariant predicates as an ambireflective subcategory of the behavioural predicates.

Example 4.4. Using this definition we can show $\triangleleft\varphi$ is always forward-invariant, i.e. $\square\triangleleft = \triangleleft$.

5 Regulation

Oftentimes, a system's goal is realized by coupling it with a second system, called a *regulator*. Formally:

Definition 5.1. Let $\pi : \mathbf{Goals} \to \mathbf{Sys}$ be a realizable goal structure. A **regulation problem** is

1. A **plant system** $\mathsf{S} : \mathbf{Sys}(J)$ with **regulation goal** $G : \mathbf{Goals}(J)(\mathsf{S})$

2. A **coupling process** $c : J \otimes I \to L$.

Regulation problems are solved by systems $\mathsf{U} : \mathbf{Sys}(I)$, called **candidate regulators**.

5.1 How to make a perfect cup of tea

Definition 5.2. A solution to a regulation problem is a **perfect regulator**: a candidate regulator that succesfully brings $\mathbf{Sys}(c)(\mathsf{U} \otimes \mathsf{S})$ to attain the goal $\mathbf{Goals}(c)(1_U \otimes G)$ induced on it. In symbols:

$$\{\mathbf{Goals}(c)(1_{\mathsf{U}^*} \otimes G)\} \cong \mathbf{Sys}(c)(\mathsf{U}^* \otimes \mathsf{S}). \tag{5.1}$$

Notice (5.1) can be naturally strengthened by asking for the isomorphism to come from the universal 2-cell witnessing the inclusion of a realizer described in Lemma 4.1. Indeed, such inclusion induces the following 2-cell:

$$\begin{array}{ccc} & \mathbf{Goals}(J) \xrightarrow{\mathbf{Goals}(c)(-\otimes G)} \mathbf{Goals}(L) & \\ {}^{1_J}\nearrow & \Downarrow \lambda_J & \searrow^{\{-\}_J} \\ \mathbf{Sys}(J) & \xrightarrow[\mathbf{Sys}(c)(-\otimes \mathsf{S})]{\qquad\qquad\qquad} & \mathbf{Sys}(L) \end{array} \tag{5.2}$$

The two 1-cells in the above diagram are taking a candidate regulator $\mathsf{U} : \mathbf{Sys}(J)$ and construction the two sides of (5.1). As observed, there always is a mapping between the results since $\pi(\mathbf{Goals}(c)(1_{\mathsf{U}^*} \otimes G)) = \mathbf{Sys}(c)(\mathsf{U}^* \otimes \mathsf{S})$.

This state of things allows us to describe perfect regulators as the objects belonging to the *inverter* [Kel89, (4.6)] of λ_J.

Indeed, the inverter of λ_J is the universal map $P \xrightarrow{i} \mathbf{Sys}(J)$ such that $i\lambda_J$ (the whiskering of λ_J by i) is invertible. In other words, is the largest subcategory of $\mathbf{Sys}(J)$ for which the components of natural transformation λ_J are invertible, hence the systems U^* satisfying (5.1).

Proposition 5.3. *The* ***solution*** *of a good regulation problem is the inverter of the corresponding diagram, depicted in* (5.2)

5.2 How to make a decent cup of tea

Most of the time regulation problems aren't solvable exactly: it might not be possible to bring S to necessarily satisfy G. Nevertheless, we are interested in singling out those Us that get as close as possible to perfect regulators. These are the **good regulators**.

The hallmark of a good regulator is not perfection but unimprovability: a good regulator V^* makes the coupled system $\{\mathbf{Goals}(c)(1_{\mathsf{V}} \otimes G)\}$ is 'as close as it can get' to $\mathbf{Sys}(c)(\mathsf{V}^* \otimes \mathsf{S})$.

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