

Representable Behaviour in Double Categorical Systems Theory

Old and new wisdom

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Double Categorical Systems Theory

DCST (Myers 2020; Myers 2021) is a principled mathematical framework for the ontology and phenomenology of systems, and distills lots of wisdom from various other categorical approaches.

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3. **Bicategories of transition systems** (Katis, Sabadini, and Walters 1997a; Katis, Sabadini, and Walters 1997b; Katis, Sabadini, and Walters 2002; Gianola, Kasangian, and Sabadini 2017; Di Lavore, Gianola, Román, Sabadini, and Sobociński 2021)

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4. **Double categories of structured cospans** (Fiadeiro and Schmitt 2007; Fong 2015; Baez and Courser 2020; Baez, Courser, and Vasilakopoulou 2022; Baez and Master 2020)

Double Categorical Systems Theory

In DCST, **systems** are organized as algebras of a symmetric double operad, or symmetric monoidal double category of **composition operations** or **processes**.

$$\begin{array}{c} \text{Sys} \\ 1 \xrightarrow{\quad} \mathbb{I} \end{array} \quad \begin{array}{c} S \\ \downarrow \varphi \\ S' \end{array} \bullet \begin{array}{ccc} I & \xrightarrow{p} & J \\ h \downarrow & \Downarrow \alpha & \downarrow k \\ I' & \xrightarrow{p'} & J' \end{array} = \begin{array}{c} S \bullet p \\ \downarrow \varphi \bullet \alpha \\ S' \bullet p' \end{array} \quad (0.1)$$

This is an etymologically accurate structure (*system* meaning 'composed of things').

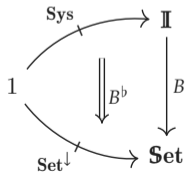
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Behaviours are then functors out of them:

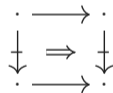
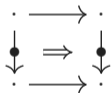


Plan of the talk

1. Theories of composition and theories of systems
 - 1.1 Composition theories as symmetric monoidal double categories
 - 1.2 Systems theories as right modules
 - 1.3 Examples: theories from adequate triples, Moore machines as free theories
2. Representable behaviour
 - 2.1 Functorial behaviour
 - 2.2 Compositionality theorem in behavioural form
 - 2.3 Multi- and plurirepresentable behaviour, nerve behaviour

Some conventions

1. Double categories are **weak** by default, (double) functors are **lax** by default
2. For the rest I mostly follow
 - ▶ M. Grandis, *Higher Dimensional Categories: From Double to Multiple Categories*. World Scientific, 2019
3. '(Loose) arrows' are marked (\dashrightarrow or $\dashv\rightarrow$), '(tight) morphisms' are not (\longrightarrow):

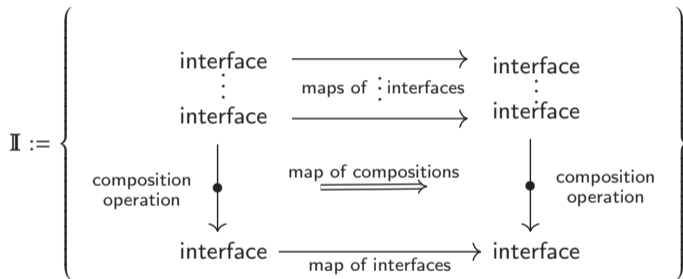


Theories of composition & theories of systems

Theories of compositions

Definition

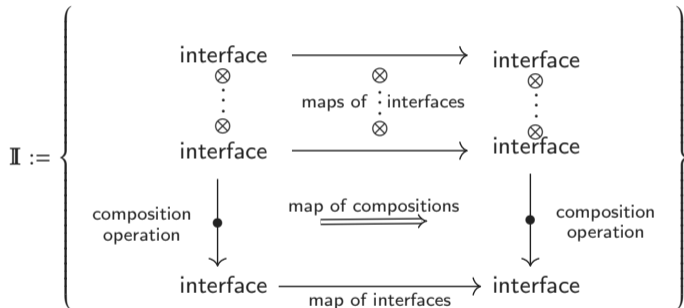
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We will assume our theories **representable**, hence **symmetric monoidal double categories**.

Theories of systems

Definition

A **theory of systems** over the theory of composition \mathbb{I} is

(tight datum) a **displayed symmetric monoidal category**, i.e. a strict monoidal isofibration:

$$\begin{array}{ccc} \mathbf{Sys} & S \xrightarrow{\varphi} S' & \\ D \downarrow & & \text{and we write } S \in \mathbf{Sys}(I), \varphi \in \mathbf{Sys}(h). \\ \mathbb{I}_0 & I \xrightarrow{h} I' & \end{array}$$

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(module structure) equipped with a (right) **module structure**, i.e. a strong monoidal functor:

$$\begin{array}{ccccc} \mathbf{Sys} & \longleftarrow & \mathbf{Sys} \times \mathbb{I}_1 & \overset{(\bullet)}{\dashrightarrow} & \mathbf{Sys} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{I}_0 & \xleftarrow{s} & \mathbb{I}_1 & \xrightarrow{t} & \mathbb{I}_0 \end{array}$$

Theories of systems

The module structure amounts to an operation

$$\begin{array}{ccc}
 \text{Sys} & \times & \mathbb{I}_1 & \xrightarrow{(\bullet)} & \text{Sys} \\
 S & I & \xrightarrow{\bullet p} & J & S \bullet p \\
 \varphi \downarrow & \bullet & h \downarrow & \theta \Downarrow & \downarrow k & = & \downarrow \varphi \bullet \theta \\
 S' & I' & \xrightarrow{\bullet p'} & J' & S' \bullet p'
 \end{array}$$

with coherent structure morphisms

unitor $S \bullet 1 \cong S,$

compositor $(S \bullet p) \bullet q \cong S \bullet (p \odot q),$

interchangers $(S \bullet p) \otimes (R \bullet q) \cong (S \otimes R) \bullet (p \otimes q)$

Example: behavioural theories

Example

For any *finitely complete category* \mathbf{E} , $\mathbf{Span}(\mathbf{E})$ is a theory of composition and $\mathbf{E} \downarrow \xrightarrow{\partial_1} \mathbf{E}$ supports a $\mathbf{Span}(\mathbf{E})$ -module structure, given by pull-push (denoted \times)

$$\begin{array}{ccc}
 S \xrightarrow{f} A & & A \xleftarrow{l} P \xrightarrow{r} B \\
 \varphi \downarrow & & \downarrow h \quad \theta \downarrow \quad \downarrow k \\
 S' \xrightarrow{f'} A' & \times & A' \xleftarrow{l'} P' \xrightarrow{r'} B'
 \end{array}
 =
 \begin{array}{ccc}
 S \times P \xrightarrow{f \times (l, r)} B & & \\
 \varphi \times \theta \downarrow & & \downarrow k \\
 S' \times P' \xrightarrow{f' \times (l', r')} B' & &
 \end{array}$$

This is the **behavioural theory** associated to \mathbf{E} . We keep denoting it as \mathbf{E} , chiefly in the case $\mathbf{E} = \mathbf{Set}$.

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Similarly, if \mathbf{E} is *regular* $\mathbf{E} \downarrow \xrightarrow{\partial_1} \mathbf{E}$ is a right module over $\mathbf{Rel}(\mathbf{E})$. We call it the **blackbox behavioural theory** associated to \mathbf{E} .

Example: adequate triples

Definition (following Haugseng, Hebestreit, Linskens, and Nuiten 2023)

A **symmetric monoidal adequate triple** is a symmetric monoidal category (\mathbf{E}, \otimes) equipped with two wide subcategories¹ whose morphisms are called *ingressive* \succrightarrow and *egressive* \searrow , such that:

1. every isomorphism is ingressive,
2. ingressive and egressive maps are closed under monoidal products,
3. every cospan as below left can be completed to a pullback as below right:

$$\begin{array}{ccc} & \cdot & \\ & \downarrow & \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \quad \mapsto \quad \begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \lrcorner & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \quad (0.2)$$

4. and \otimes commutes with ingressive-egressive pullbacks,

i.e.: $E \downarrow \xrightarrow{\partial_1} E$ is a strict monoidal isofibration admitting strong cartesian lifts of every ingressive map.

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For every symmetric monoidal adequate triple $(\mathbf{E}, \succ, \twoheadrightarrow)$, $\mathbf{Span}(\mathbf{E})$ is a theory of composition and $\mathbf{E} \downarrow \xrightarrow{\partial_1} \mathbf{E}$ supports a $\mathbf{Span}(\mathbf{E}, \succ, \twoheadrightarrow)$ -module structure:

$$\begin{array}{ccc}
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 \end{array} & \times & \begin{array}{ccc}
 A & \xleftarrow{l} & P & \xrightarrow{r} & B \\
 h \downarrow & & \theta \downarrow & & \downarrow k \\
 A' & \xleftarrow{l'} & P' & \xrightarrow{r'} & B'
 \end{array} \\
 & & = & & \begin{array}{ccc}
 S \times P & \xrightarrow{f \times (l, r)} & B \\
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Example: P -charts & P -lenses

Let $P : \mathbf{E} \rightarrow \mathbf{B}$ be a strict symmetric monoidal fibration. We represent \mathbf{E} as P -charts:

$$\begin{pmatrix} A^- \\ A^+ \end{pmatrix} \begin{array}{c} \xrightarrow{h^b} \\ \xrightarrow{h} \end{array} \begin{pmatrix} A'^- \\ A'^+ \end{pmatrix} = \begin{pmatrix} A^- \\ A^+ \end{pmatrix} \begin{array}{c} \xrightarrow{\begin{pmatrix} h^b \\ A^+ \end{pmatrix}} \\ \xrightarrow{\quad} \end{array} \begin{pmatrix} A'^- \\ A^+ \end{pmatrix} \begin{array}{c} \xrightarrow{\begin{pmatrix} A'^- \\ h \end{pmatrix}} \\ \xrightarrow{\quad} \end{array} \begin{pmatrix} A'^- \\ A'^+ \end{pmatrix}$$

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Turns out $(\mathbf{E}, \text{vert}, \text{cart})$ is a symmetric monoidal adequate triple, thus we can define:

$$\mathbf{Span}(P) := \mathbf{Span}(\mathbf{E}, \text{vert}, \text{cart}) = \left\{ \begin{array}{ccc} & \begin{matrix} \xrightarrow{\begin{pmatrix} h^b \\ h \end{pmatrix}} \\ \xrightarrow{\quad} \end{matrix} & \\ \begin{pmatrix} A^- \\ A^+ \end{pmatrix} & & \begin{pmatrix} A'^- \\ A'^+ \end{pmatrix} \\ \begin{pmatrix} f^\sharp \\ A^+ \end{pmatrix} \parallel\uparrow & \begin{matrix} \xrightarrow{\begin{pmatrix} \theta^b \\ \theta \end{pmatrix}} \\ \xrightarrow{\quad} \end{matrix} & \parallel\uparrow \begin{pmatrix} f'^\sharp \\ A'^+ \end{pmatrix} \\ \begin{pmatrix} B^- \\ A^+ \end{pmatrix} & & \begin{pmatrix} B'^- \\ A'^+ \end{pmatrix} \\ \begin{pmatrix} B^- \\ f \end{pmatrix} \parallel\downarrow & \begin{matrix} \xrightarrow{\begin{pmatrix} k^b \\ k \end{pmatrix}} \\ \xrightarrow{\quad} \end{matrix} & \parallel\downarrow \begin{pmatrix} B^- \\ f' \end{pmatrix} \\ \begin{pmatrix} B^- \\ B^+ \end{pmatrix} & & \begin{pmatrix} B'^- \\ B'^+ \end{pmatrix} \end{array} \right\}$$

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Example: P -charts & P -lenses

We denote a span as above with the shorter notation:

$$\begin{pmatrix} A^- \\ A^+ \end{pmatrix} \begin{matrix} \xleftarrow{f^\#} \\ \xrightarrow{f} \end{matrix} \begin{pmatrix} B^- \\ B^+ \end{pmatrix} := \begin{pmatrix} A^- \\ A^+ \end{pmatrix} \begin{matrix} \xleftarrow{\begin{pmatrix} f^\# \\ A^+ \end{pmatrix}} \\ \xrightarrow{\begin{pmatrix} B^- \\ f \end{pmatrix}} \end{matrix} \begin{pmatrix} B^- \\ B^+ \end{pmatrix}$$

This is a P -**lens** (Spivak 2022; Capucci, Gavranović, Malik, Rios, and Weinberger 2024).

The category of P -lenses associated to $\mathbf{Set} \downarrow \xrightarrow{\partial_1} \mathbf{Set}$ is equivalent to **Poly** (Niu and Spivak 2023).

Example: P -charts & P -lenses

Therefore, P -charts and P -lenses form a thin double category $\mathbf{Lens}(P) \equiv \mathbf{Span}(P)$, whose squares are as above and denoted as below:

$$\begin{array}{ccc}
 \left(\begin{array}{c} A^- \\ A^+ \end{array} \right) & \begin{array}{c} \xrightarrow{h^b} \\ \xrightarrow{h} \end{array} & \left(\begin{array}{c} A'^- \\ A'^+ \end{array} \right) \\
 \begin{array}{c} \downarrow f \\ \uparrow f^\sharp \end{array} & & \begin{array}{c} \downarrow f' \\ \uparrow f'^\sharp \end{array} \\
 \left(\begin{array}{c} B^- \\ B^+ \end{array} \right) & \begin{array}{c} \xrightarrow{k^b} \\ \xrightarrow{k} \end{array} & \left(\begin{array}{c} B'^- \\ B'^+ \end{array} \right)
 \end{array}$$

In type-theoretic notation, these encode the following commutativity condition:

$$\begin{array}{l}
 \forall a^+ : A^+, \quad k(f(a^+)) = f'(h(a^+)), \\
 a^+ : A^+ \vdash \forall b^- : B^-(f(a^+)), \quad h^b(a^+, f^\sharp(a^+, b^-)) = f'^\sharp(h(a^+), k^b(f(a^+), b^-)).
 \end{array} \tag{0.3}$$

Example: Moore machines

On $\mathbf{Lens}(\mathbf{Set} \downarrow \xrightarrow{\partial_1} \mathbf{Set})$ we consider two different theories of systems:

- deterministic discrete Moore machines** $\mathbf{Moore}(\mathbf{Set})$, where a Moore machine over $\begin{pmatrix} I \\ O \end{pmatrix}$ is a lens as below left and a morphism of Moore machines is a map φ (over the chart $\begin{pmatrix} h^b \\ h \end{pmatrix}$) making the square below commute:

$$\begin{pmatrix} S \\ S \end{pmatrix} \begin{array}{c} \xrightarrow{v^\#} \\ \xleftarrow{v} \end{array} \begin{pmatrix} I \\ O \end{pmatrix} \equiv \begin{cases} v : S \rightarrow O, \\ v^\# : (s : S) \times I(v(s)) \rightarrow S \end{cases}$$

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & S' \\ \begin{pmatrix} S \\ S \end{pmatrix} & \begin{array}{c} \xrightarrow{\varphi\pi_2} \\ \xrightarrow{\varphi} \end{array} & \begin{pmatrix} S' \\ S' \end{pmatrix} \\ \begin{array}{c} \downarrow v \\ \uparrow v^\# \end{array} & & \begin{array}{c} \downarrow v' \\ \uparrow v^{\#'} \end{array} \\ \begin{pmatrix} I \\ O \end{pmatrix} & \begin{array}{c} \xrightarrow{h^b} \\ \xrightarrow{h} \end{array} & \begin{pmatrix} I' \\ O' \end{pmatrix} \end{array}$$

The module structure is given by composition of lenses and (looseward) composition of squares.

Example: Moore machines

On $\mathbf{Lens}(\mathbf{Set} \downarrow \xrightarrow{\partial_1} \mathbf{Set})$ we consider two different theories of systems:

1. **deterministic discrete Moore machines** $\mathbf{Moore}(\mathbf{Set})$
2. **possibilistic discrete Moore machines** $\mathbf{Moore}_{\mathcal{P}}(\mathbf{Set})$ are similarly defined, except now a Moore machine is given by a non-deterministic lens; while a map is square which commutes only up to containment:

$$\begin{pmatrix} S \\ S \end{pmatrix} \begin{matrix} \xrightarrow{v^\#} \\ \xleftarrow{v} \end{matrix} \begin{pmatrix} I \\ O \end{pmatrix} \equiv \begin{cases} v : S \rightarrow O, \\ v^\# : (s : S) \times I(v(s)) \rightarrow \mathcal{P}S \end{cases}$$

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Intuitively: the transitions out of $\varphi(s) \in S'$ must contain *at least* the image of those out of $s \in S$.

Example: free theories

Given a displayed symmetric monoidal category $T : \mathbf{X} \rightarrow \mathbb{I}_0$, the **free theory on T** is

$T \not\circ \mathbb{I} := \mathbb{I}[T] := T \times \mathbb{I}$:

$$\begin{array}{ccccc}
 \mathbf{X} \times \mathbb{I}_1 & \longleftarrow & \mathbf{X} \times \mathbb{I}_1 \times \mathbb{I}_1 & \overset{\mathbf{X} \times (\odot)}{\dashrightarrow} & \mathbf{X} \times \mathbb{I}_1 \\
 \downarrow T \times \mathbb{I} & & \downarrow T \times \mathbb{I} & & \downarrow T \times \mathbb{I} \\
 \mathbb{I}_0 & \xleftarrow{s} & \mathbb{I}_1 & \xrightarrow{t} & \mathbb{I}_0
 \end{array}$$

Systems over $J \in \mathbb{I}$ are given by 'formal composites' of generators $G \in \mathbf{X}(I)$ and a process $I \xrightarrow{p} J$.

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Example

Given a section $T : \mathbf{B} \rightarrow \mathbf{E}$ of a fibration $P : \mathbf{E} \rightarrow \mathbf{B}$, the free theory $T \blacklozenge \mathbf{Lens}(P)$ is the **theory of (generalized) Moore machines** (this construction is central in (Myers 2021)):

$$\underbrace{\begin{pmatrix} TS \\ S \end{pmatrix}}_{\text{generator}} \overset{v^\#}{\underset{v}{\rightleftarrows}} \begin{pmatrix} I \\ O \end{pmatrix}.$$

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Notable instances are: open ODEs being free on T a tangent structure on \mathbf{B} , $\mathbf{Moore}(\mathbf{Set})$ being free on $S \xrightarrow{T} S \times S \xrightarrow{\pi_1} S$.

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Given a displayed symmetric monoidal category $T : \mathbf{X} \rightarrow \mathbb{I}_0$, the **free theory on T** is

$$T \blacklozenge \mathbb{I} := \mathbb{I}[T] := T \times \mathbb{I}:$$

$$\begin{array}{ccccc} \mathbf{X} \times \mathbb{I}_1 & \longleftarrow & \mathbf{X} \times \mathbb{I}_1 \times \mathbb{I}_1 & \overset{\mathbf{X} \times (\odot)}{\dashrightarrow} & \mathbf{X} \times \mathbb{I}_1 \\ \downarrow T \times \mathbb{I} & & \downarrow T \times \mathbb{I} & & \downarrow T \times \mathbb{I} \\ \mathbb{I}_0 & \xleftarrow{s} & \mathbb{I}_1 & \xrightarrow{t} & \mathbb{I}_0 \end{array}$$

Systems over $J \in \mathbb{I}$ are given by 'formal composites' of generators $G \in \mathbf{X}(I)$ and a process $I \xrightarrow{p} J$.

Example

Given a section $T : \mathbf{B} \rightarrow \mathbf{E}$ of a fibration $P : \mathbf{E} \rightarrow \mathbf{B}$, the free theory $T \blacklozenge \mathbf{Lens}(P)$ is the **theory of (generalized) Moore machines** (this construction is central in (Myers 2021)):

$$\underbrace{\begin{pmatrix} TS \\ S \end{pmatrix}}_{\text{generator}} \overset{v^\#}{\underset{v}{\rightleftarrows}} \begin{pmatrix} I \\ O \end{pmatrix}.$$

Notable instances are: open ODEs being free on T a tangent structure on \mathbf{B} , $\mathbf{Moore}(\mathbf{Set})$ being free on $S \xrightarrow{T} S \times S \xrightarrow{\pi_1} S$. **Beware!** $\mathbf{Moore}_P(\mathbf{Set})$ is not free but it's subfree.

Functorial behaviour

Functorial behaviour

Idea: while *theories of systems* describe the structural (morphological & compositional) aspects of systems, *functors* out of them describe their **behavioural/dynamical** aspects:

$$B : \mathbf{Sys} \rightarrow \mathbf{E}$$

Usually, the codomain is a (you guessed it) behavioural theory.

This is a form of **functorial semantics**, since the functor itself establishes a relationship between two theories in which the domain is 'interpreted' in the codomain.

Morphisms of systems theories

Definition

A **lax morphism of systems theories** $\begin{pmatrix} F^b \\ F \end{pmatrix} : \begin{pmatrix} \text{Sys} \\ \mathbb{I} \end{pmatrix} \rightarrow \begin{pmatrix} \text{Sys}' \\ \mathbb{I}' \end{pmatrix}$ is given by

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$$\mathbb{I} \xrightarrow{F} \mathbb{I}' \qquad \mathbf{Sys}(I) \xrightarrow{F_I^b} \mathbf{Sys}'(FI)$$

(laxators) and suitably coherent laxators as below:

$$\text{monoidal laxators} \quad 1' \xrightarrow{v} F1, \quad FI \otimes' FJ \xrightarrow{v} F(I \otimes J)$$

$$1' \xrightarrow{v^b} F^b 1, \quad F^b(S) \otimes' F^b(R) \xrightarrow{v^b} F^b(S \otimes R)$$

$$\text{compositional laxators} \quad 1' \xrightarrow{\eta} F1, \quad Fp \odot' Fq \xrightarrow{\kappa} F(p \odot q)$$

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$\left(\begin{smallmatrix} F^b \\ F \end{smallmatrix} \right)$ is **strong monoidal/compositional** when the corresponding laxators are invertible.

Theory of behaviour

Definition

A **theory of behaviour** $\left(\begin{smallmatrix} B^b \\ B \end{smallmatrix} \right) : \left(\begin{smallmatrix} \mathbf{Sys} \\ \mathbb{I} \end{smallmatrix} \right) \rightarrow \left(\begin{smallmatrix} \mathbf{Set}^\downarrow \\ \mathbf{Set} \end{smallmatrix} \right)$ is given by

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(part on systems) a displayed symmetric monoidal functor as below right,

$$\mathbb{I} \xrightarrow{B} \mathbf{Set} \qquad \mathbf{Sys}(I) \xrightarrow{B_I^b} \mathbf{Set}/BI$$

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Aside: obstructions to compositionality

One can classify obstructions to monoidality/compositionality by factoring the laxators, e.g. for ℓ :

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the whole exhibits new behaviours.
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Blackboxing $\blacksquare : \mathbf{Set} \rightarrow \mathbf{Rel}$ ignores internals and thus ‘localizes’ behaviour at the 0-truncated behaviour types, focusing on missing behaviours (as done e.g. in (Master 2021))

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Very general idea! Works for any finitely complete category \mathbf{E} equipped with a *modality* \blacksquare , e.g. a lex reflective subcategory.

Representable behaviour

Representability allows to tame the complexity of a theory of behaviour & it is very common in nature.

Definition

A **representable theory of behaviour** over \mathbf{Sys} is one given by

$$\left(\begin{array}{c} \mathbf{Sys}(C, -) \\ \mathbb{I}(H, -) \end{array} \right)$$

for some **commutative comonoidal system** $C \in \mathbf{Sys}(H)$.

We think of C as a **clock**, with interface H being its '**hands**'.

Representable behaviour

On interfaces, $\text{Sys}(C, -)$ is given by the Parè representable at $\mathbb{I}(H, -) : \mathbb{I} \rightarrow \mathbf{Set}$:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 I & \xrightarrow{h} & I' \\
 p \downarrow & \xRightarrow{\theta} & \downarrow p' \\
 J & \xrightarrow{k} & J'
 \end{array} & \mapsto & \begin{array}{ccc}
 \mathbb{I}(H, I) & \xrightarrow{h_*} & \mathbb{I}(H, I') \\
 \text{top} \uparrow & & \uparrow \text{top} \\
 \left\{ \begin{array}{ccc}
 H & \longrightarrow & I \\
 \parallel & \xRightarrow{\quad} & \downarrow p \\
 H & \longrightarrow & J
 \end{array} \right\} & \xrightarrow{\theta_*} & \left\{ \begin{array}{ccc}
 H & \longrightarrow & I' \\
 \parallel & \xRightarrow{\quad} & \downarrow p' \\
 H & \longrightarrow & J'
 \end{array} \right\} \\
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 \end{array}
 \end{array}$$

The comonoid structure (ε, Δ) on H defines the monoidal laxators:

$$1 \xrightarrow{\text{id}_1} \mathbb{I}(1, 1) \xrightarrow{\varepsilon^*} \mathbb{I}(H, 1), \quad \mathbb{I}(H, I) \times \mathbb{I}(H, J) \xrightarrow{(\otimes)} \mathbb{I}(H \otimes H, I \otimes J) \xrightarrow{\Delta^*} \mathbb{I}(H, I \otimes J)$$

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 I \xrightarrow{h} I' & \xrightarrow{\theta} & \left\{ \begin{array}{ccc} H & \longrightarrow & I \\ \bullet & \Downarrow & \bullet \\ H & \longrightarrow & J \end{array} \right\} \xrightarrow{\theta_*} \left\{ \begin{array}{ccc} H & \longrightarrow & I' \\ \bullet & \Downarrow & \bullet \\ H & \longrightarrow & J' \end{array} \right\} \\
 p \downarrow & \Downarrow & \downarrow p' \\
 J \xrightarrow{k} J' & & \\
 & & \downarrow \text{bottom} \qquad \qquad \qquad \downarrow \text{bottom} \\
 & & \mathbb{I}(H, J) \xrightarrow{k_*} \mathbb{I}(H, J')
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The compositional laxators are induced by looseward identity/composition of squares.

Representable behaviour

Similarly, on systems, we get a functor $\mathbf{Sys}(\mathbf{C}, -) : \mathbf{Sys} \rightarrow \mathbf{Set}/\mathbb{I}(H, -)$.

$$S \xrightarrow{\varphi} S' \mapsto \begin{array}{ccc} \mathbf{Sys}(\mathbf{C}, S) & \xrightarrow{\varphi_*} & \mathbf{Sys}(\mathbf{C}, S') \\ D \downarrow & & \downarrow D \\ \mathbb{I}(H, I) & \xrightarrow{(D\varphi)_*} & \mathbb{I}(H, I') \end{array}$$

Again, the comonoid structure of \mathbf{C} induces monoidal laxators, and the compositional laxators are given by composition:

$$\mathbf{Sys}(\mathbf{C}, S) \times \mathbb{I}(H, p) \xrightarrow{\ell} \mathbf{Sys}(\mathbf{C}, S \bullet p)$$

$$\begin{array}{ccc} \begin{array}{c} \mathbf{C} \\ \varphi \downarrow \\ S \end{array} & \begin{array}{c} H \bullet H \\ h \downarrow \quad \theta \Downarrow \quad k \downarrow \\ I \xrightarrow{p} J \end{array} & \mapsto \begin{array}{c} \mathbf{C} \\ \Downarrow \eta \\ \mathbf{C} \bullet 1_H \\ \downarrow \varphi \bullet \theta \\ S \bullet p \end{array} \end{array}$$

Representable behaviour for non-/deterministic Moore machines

Example

The theory of trajectories is representable by the walking trajectory

$$T_\omega := 0 \rightsquigarrow 1 \rightsquigarrow 2 \rightsquigarrow \dots$$

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The theory of fixpoints is representable by the walking fixpoint $L_1 := 0 \text{ } \text{\textcircled{+}}$.

Similarly, the theory of n -th order cycles are represented by walking loops $L_n \in \mathbf{MooreP} \left(\begin{smallmatrix} 1 \\ n \end{smallmatrix} \right)$.

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Similarly, the theory of n -th order cycles are represented by walking loops $L_n \in \mathbf{Moore}_{\mathcal{P}}\left(\begin{smallmatrix} 1 \\ n \end{smallmatrix}\right)$.

Non-example

The theory of states $\mathbf{Moore}_{\mathcal{P}}\left(\begin{smallmatrix} I \\ O \end{smallmatrix}\right) \ni S \xrightarrow{\varphi} S' \mapsto S \xrightarrow{\varphi} S \in \mathbf{Set}/1$ is represented by the initial system $0 \in \mathbf{Moore}_{\mathcal{P}}\left(\begin{smallmatrix} \emptyset \\ \emptyset \end{smallmatrix}\right)$.

States *with observations* are represented by $L_0 := 0$.

Compositionality of representable behaviours

Compositionality of representable behaviours hinges on three properties:

1. \mathbb{I} and \mathbf{Sys} are **cartesian**, in which case

$$\begin{array}{l} C \xrightarrow{-\exists!} 1 \\ H \xrightarrow{-\exists!} 1 \end{array} \quad \begin{array}{l} C \rightarrow S \times R \\ H \rightarrow I \times J \end{array} = \begin{pmatrix} C \xrightarrow{-\exists!} S & C \xrightarrow{-\exists!} R \\ H \xrightarrow{-\exists!} I' & H \xrightarrow{-\exists!} J \end{pmatrix}$$

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1. \mathbb{I} and \mathbf{Sys} are **cartesian**, in which case **the monoidal laxators are invertible**:

$$1 \xrightarrow{\sim} \mathbf{Sys}(C, 1),$$

$$\mathbf{Sys}(C, S) \times \mathbf{Sys}(C, R) \xrightarrow{\sim} \mathbf{Sys}(C, S \times R)$$

$$1 \xrightarrow{\sim} \mathbb{I}(H, 1),$$

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This is more common than it looks: **all the examples we mentioned so far are cartesian.**

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2. \mathbb{I} is **spanlike**, in which case

$$\begin{array}{ccc} H & \longrightarrow & I \\ \parallel & \xRightarrow{\exists!} & \parallel \\ H & \longrightarrow & I \end{array}$$

$$\begin{array}{ccc} H & \longrightarrow & I \\ \parallel & & \downarrow p \\ \bullet & \Longrightarrow & J \\ \parallel & & \downarrow q \\ H & \longrightarrow & K \end{array} = \begin{array}{ccc} H & \longrightarrow & I \\ \parallel & \xRightarrow{\exists!} & \downarrow p \\ H & \xrightarrow{\exists!} & J \\ \parallel & \xRightarrow{\exists!} & \downarrow q \\ H & \longrightarrow & K \end{array}$$

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$$1 \xrightarrow{\sim} \mathbb{I}(H, 1), \quad \mathbb{I}(H, p) \times \mathbb{I}(H, q) \xrightarrow{\sim} \mathbb{I}(H, p \odot q)$$

This too is the case for **all the composition theories we mentioned so far** (because they are literally double categories of spans).

Compositionality of representable behaviours

3. Sys is **observable**, in which case

$$\begin{array}{c} C \xrightarrow{\varphi} S \bullet p \\ H \xrightarrow{k} J \end{array} = \begin{array}{c} C \xrightarrow{\exists!} S \\ \bullet \\ H \xrightarrow{\exists!} I \\ \Downarrow \exists! \downarrow p \\ H \xrightarrow{k} J \end{array}$$

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This is rarely the case!

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This is rarely the case!

All the above properties need not hold for the entirety of \mathbf{Sys} and \mathbf{I} , it's enough they hold 'at $\begin{pmatrix} C \\ H \end{pmatrix}$ ':

Definition

We say $\begin{pmatrix} C \\ H \end{pmatrix}$ is **cartesian/spanlike/observable** when the corresponding laxators for the representable $\begin{pmatrix} C \\ H \end{pmatrix}$ are invertible.

Observability of a system in \mathbf{E}^\downarrow

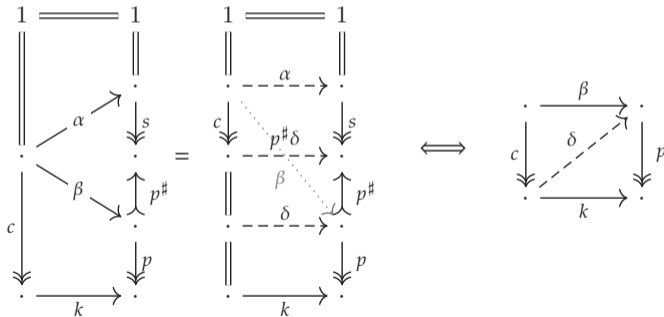
Let \mathbf{E}^\downarrow be the theory associated to an adequate triple. It is cartesian and spanlike, but in general not observable.

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Lemma

The factorization problem on the left is equivalent to the lifting problem on the right:

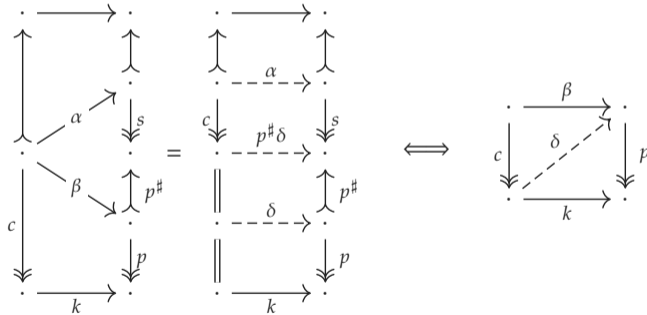


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Observability of a system in $\mathbf{E} \Downarrow$

Theorem

Let \mathbf{E} be a symmetric monoidal adequate triple, $T : \mathbf{X} \rightarrow \mathbf{E}$ displayed symmetric monoidal category.

A system $TC \xleftarrow{c^\#} \cdot \xrightarrow{c} H$ is observable in the free theory $T \nabla \mathbf{Span}(\mathbf{E})$ iff c is left orthogonal to all egressive maps:



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Corollary (\Leftarrow is Myers 2021, Theorem 5.3.3.1)

For (P, T) theory of Moore machines, recall $T \blacklozenge \mathbf{Span}(P) = \mathbf{Moore}(P, T)$, thus $\begin{pmatrix} TC \\ C \end{pmatrix} \xrightleftharpoons[c]{c^\#} \begin{pmatrix} H^- \\ H^+ \end{pmatrix}$ is observable iff c is invertible.

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Corollary

If c is split epi, $\begin{pmatrix} TC \\ C \end{pmatrix} \xrightleftharpoons[c]{c^\#} \begin{pmatrix} H^- \\ H^+ \end{pmatrix}$ induces surjective laxators, i.e. **there are no 0-generative effects**.

Multirepresentable behaviour

More often than not, probing from a *single* system C doesn't cut it, instead behaviours comes in various shapes, each of which needs its own separate archetypal system:

$$B = \sum_{t \in T} \mathbf{Sys}(C_t, -)$$

for a *family* $C : T \rightarrow \mathbf{Sys}$ (i.e. indexed by a set T).

Such functors are called **multirepresentable** (Karazeris and Velebil 2009).

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Remark

The family C is intended to be a *colax map of systems theories* from an **americ discrete theory** T :

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However, the coproduct is take in the displayed category $[\mathbf{Sys}, \mathbf{Set}] \rightarrow [\mathbf{I}, \mathbf{Set}]$.

Multirepresentable behaviour: non-deterministic Moore machines

Example

The simplest case of multirepresentable behaviour is that of **runs** (or **paths**) of Moore machines. In that case we have

$$\begin{array}{ccc} \mathbf{N} & \xrightarrow{\quad \mathsf{T} \quad} & \mathbf{Moore}_{\mathcal{P}}(\mathbf{Set}) \\ n & \mapsto & 0 \rightsquigarrow \dots \rightsquigarrow n \end{array}$$

where we stress T_n has interface $ny + 1$.

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$$\mathbf{N} \xrightarrow{\mathbf{T}} \mathbf{Moore}_{\mathcal{P}}(\mathbf{Set})$$
$$n \mapsto 0 \rightsquigarrow \dots \rightsquigarrow n$$

where we stress \mathbf{T}_n has interface $ny + 1$.

On interfaces, a map $\mathbf{T}_n \rightarrow \mathbf{S}$ corresponds to a choice of $n + 1$ outputs and n compatible inputs:

$$\binom{n}{n+1} \Rightarrow \binom{I}{O} \quad \Leftrightarrow \quad \{((o_0, \dots, o_n), (i_1, \dots, i_n)) \mid i_{k+1} \in O_k \text{ for } 0 \leq k < n\}$$

and on systems, to a suitable sequence of n transitions $s_0 \rightsquigarrow^{i_1} s_1 \rightsquigarrow^{i_2} \dots \rightsquigarrow^{i_n} s_n$ such that $v(s_k) = o_k$.

Multirepresentable behaviour: non-deterministic Moore machines

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Example

Likewise, the family \mathbf{L}_n described before multirepresents the theory of loops.

Compositionality of multirepresentable behaviour

To construct the monoidal laxators, we need the family $C : T \rightarrow \mathbf{Sys}$ to be *colax monoidal*, thus

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In any case we get (analogously on \mathbb{I}):

$$\sum_t \mathbf{Sys}(C_t, S) \times \sum_s \mathbf{Sys}(C_s, R) \xrightarrow{\sim} \sum_{t,s} \mathbf{Sys}(C_t, S) \times \mathbf{Sys}(C_s, R) \xrightarrow{\sum_{t,s} \nu_{s,t}} \sum_{t,s} \mathbf{Sys}(C_{t \otimes s}, S \otimes R) \xrightarrow{\sum_{\otimes}} \sum_t \mathbf{Sys}(C_t, S \otimes R)$$

We don't have much control over \sum_{\otimes} (invertible when $(t, s) \mapsto t \otimes s$ is—rarely).

Note: this is a pointwise coproduct but not a coproduct in $[\mathbf{Sys}, \mathbf{Set}]$!

Compositionality of multirepresentable behaviour

The absence of non-trivial loose arrows in the indexing T makes the compositional laxators analogous to the simply representable situation:

$$\begin{aligned} \sum_{t \in T} \mathbf{I}(H_t, p) \times \sum_{t \in T} \mathbf{I}(H_t, q) &\cong \sum_{t \in T} \mathbf{I}(H_t, p) \times \mathbf{I}(H_t, q) \xrightarrow{\sum_{t \in T} (\Xi)} \sum_{t \in T} \mathbf{I}(H_t, p \odot q) \\ \sum_{t \in T} \mathbf{Sys}(C_t, S) \times \sum_{t \in T} \mathbf{I}(H_t, p) &\cong \sum_{t \in T} \mathbf{Sys}(C_t, S) \times \mathbf{I}(H_t, p) \xrightarrow{\sum_{t \in T} (\bullet)} \sum_{t \in T} \mathbf{Sys}(C_t, S \bullet p) \end{aligned}$$

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Theorem

The behaviour multirepresented by $C : T \rightarrow \mathbf{Sys}$ is strongly compositional iff

1. *each H_t is spanlike,*
2. *each C_t is observable.*

Plurirepresentable behaviour

The family of systems we want to use to induce a theory of behaviour are not unrelated to each other, and thus rather than a multirepresentable functor we get a **plurirepresentable** one:

$$B = \operatorname{colim}_{t \in \mathbf{T}} \mathbf{Sys}(C_t, -)$$

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The morphisms of \mathbf{T} make this situation particularly interesting, since they witness a 'geometric structure' on timepieces (especially when \mathbf{T} is endowed with a coverage).

Plurirepresentable behaviour: non-deterministic Moore machines

Example

The family of loops L_n can be indexed by $(\mathbb{N}, |)^{\text{op}}$, since a loop L_n can be wound up around a loop L_m only if $m \mid n$. Then $\text{colim}_n L_n S$ yields the **minimal/indecomposable loops** in S and thus is less redundant than $\sum_n L_n S$. One can do better by categorifying...

Plurirepresentable behaviour: non-deterministic Moore machines

Non-example

A natural plurirepresenter candidate for **maximal runs** of Moore machines is

$$\begin{array}{c} (\mathbb{N}, \leq) \longrightarrow \mathbf{Moore}_{\mathcal{P}}(\mathbf{Set}) \\ \begin{array}{c} n \\ \vee \wedge \\ m \end{array} \longmapsto \begin{array}{ccccc} 0 & \longrightarrow & \cdots & \longrightarrow & n \\ \vdots & & \vdots & & \vdots \\ 0 & \longrightarrow & \cdots & \longrightarrow & n \longrightarrow \cdots \longrightarrow m \end{array} \end{array}$$

However, $\operatorname{colim}_{n \in \mathbb{N}} \mathbf{Moore}_{\mathcal{P}}([n], -)$ just yields the states of the machine, since every runs gets identified with its prefixes.

Still, the behaviour is plurirepresentable since it is multirepresentable (one needs to stick to *the set* \mathbb{N}).

Compositionality of plurirepresentable behaviour

Plurirepresentable behaviour has much of the same problems regarding monoidality as multirepresentable behaviour.

As for compositionality, we now need to assume \mathbf{T} is **cofiltered** to get the right distributivity of colimit and pullback (since the colimit is indexed by \mathbf{T}^{op}):

$$\begin{aligned} \text{colim}_{t \in \mathbf{T}} \mathbb{I}(H_t, p) \times \text{colim}_{t \in \mathbf{T}} \mathbb{I}(H_t, q) &\xrightarrow{\sim} \text{colim}_{t \in \mathbf{T}} \mathbb{I}(H_t, p) \times \mathbb{I}(H_t, q) \xrightarrow{\text{colim}_{t \in \mathbf{T}} (\Xi)} \text{colim}_{t \in \mathbf{T}} \mathbb{I}(H_t, p \odot q) \\ \text{colim}_{t \in \mathbf{T}} \mathbf{Sys}(C_t, S) \times \text{colim}_{t \in \mathbf{T}} \mathbb{I}(H_t, p) &\xrightarrow{\sim} \text{colim}_{t \in \mathbf{T}} \mathbf{Sys}(C_t, S) \times \mathbb{I}(H_t, p) \xrightarrow{\text{colim}_{t \in \mathbf{T}} (\bullet)} \text{colim}_{t \in \mathbf{T}} \mathbf{Sys}(C_t, S \bullet p) \end{aligned}$$

e.g. $(\mathbb{N}, |)$ is cofiltered.

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Theorem

The behaviour plurirepresented by $C : \mathbf{T} \rightarrow \mathbf{Sys}$ is strongly compositional iff

1. \mathbf{T} is cofiltered,
2. each H_t is spanlike,
3. each C_t is observable.

Better behaved behaviour: nerve behaviour

As we have seen, colimits can trivialize behaviour. Thus given $C : \mathbf{T} \rightarrow \mathbf{Sys}$, we get a better notion of behaviour in \mathbf{T} -variable sets by a **nerve construction**:

$$\mathbf{Sys} \xrightarrow{\mathbf{Sys}(C_{(-)}, -)} \mathbf{Set}^{\mathbf{T}^{\text{op}}}$$

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Plurirepresentable behaviour can then be obtained by taking colimits:

$$\begin{array}{ccc} \mathbf{Sys} & \xrightarrow{\text{colim}_{t \in \mathbf{T}} \mathbf{Sys}(C_t, -)} & \mathbf{Set} \\ & \searrow N_C := \mathbf{Sys}(C_{(-)}, -) & \nearrow \text{colim}_{t \in \mathbf{T}} - \\ & & \mathbf{Set}^{\mathbf{T}^{\text{op}}} \end{array}$$

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But one can use other functors, like global sections (itself a representable behaviour!)

$$\begin{array}{ccc} \mathbf{Sys} & \xrightarrow{\Gamma \mathbf{Sys}(C_{(-)}, -)} & \mathbf{Set} \\ & \searrow \mathbf{Sys}(C_{(-)}, -) & \nearrow \Gamma \\ & & \mathbf{Set}^{\mathbf{T}^{\text{op}}} \end{array}$$

e.g. the behaviour of **infinite trajectories** of non-deterministic Moore machines.

Better behaved behaviour: nerve behaviour

Now considering $\text{Set}^{\mathbf{T}^{\text{op}}}$ with

1. ...pointwise cartesian products N_C **is strong monoidal iff each C_t is cartesian.**
2. ...**Day convolution** induced by the monoidal product of \mathbf{T} , N_C **is strong monoidal as soon as C is.**

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The nerve behaviour $\text{Sys}(C_{(-)}, -)$ induced by $C : \mathbf{T} \rightarrow \mathbf{Sys}$ is strong monoidal (wrt Day) and compositional iff

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Note that:

1. for plurirepresentable behaviours, we isolated away the issue of cofilteredness,
2. since $\Gamma = \mathbf{Set}^{\mathbf{T}^{\text{op}}}(1, -)$ is compositional (by the first compositionality theorem), then we the above yields a **new class of compositionality theorem for 'semi-representable' behaviours** (those that factor as the global sections of a nerve).

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 - e.g. nerves of Moore machines are Segal in the traditional sense
 - e.g. in (Myers 2021, §6.2.3) Myers shows that nerves of ODEs are Segal in the sense of (Schultz and Spivak 2019).

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3. Maps of timepieces (*time extensions*) can be used to define categorically the behavioural properties of systems.

The most famous example is (Joyal, Nielsen, and Winskel 1996) using time-injective maps to define bisimulation. In (Baltieri, Biehl, Capucci, and Virgo 2025) we give a definition of 'model of a system' based on this.

Thanks!

References I

- [1] J. C. Baez and K. Courser, “Structured cospans”, *Theory and Applications of Categories*, vol. 35, no. 48, pp. 1771–1822, 2020.
- [2] J. C. Baez and J. Master, “Open petri nets”, *Mathematical Structures in Computer Science*, vol. 30, no. 3, pp. 314–341, 2020.
- [3] J. C. Baez, K. Courser, and C. Vasilakopoulou, “Structured versus Decorated Cospans”, *Compositionality*, vol. 4, 3 Sep. 2022, ISSN: 2631-4444. DOI: 10.32408/compositionality-4-3. [Online]. Available: <https://doi.org/10.32408/compositionality-4-3>.
- [4] P. Baldan, F. Bonchi, H. Kerstan, and B. König, “Coalgebraic behavioral metrics”, *Logical Methods in Computer Science*, vol. 14, 2018. [Online]. Available: <https://lmcs.episciences.org/4827/pdf> (visited on 11/15/2023).
- [5] M. Baltieri, M. Biehl, M. Capucci, and N. Virgo, “A Bayesian Interpretation of the Internal Model Principle”, arXiv: 2503.00511 [math]. (Mar. 1, 2025), [Online]. Available: <http://arxiv.org/abs/2503.00511> (visited on 03/06/2025), pre-published.

References II

- [6] M. Capucci, B. Gavranović, A. Malik, F. Rios, and J. Weinberger, “On a fibrational construction for optics, lenses, and Dialectica categories”, *Electronic Notes in Theoretical Informatics and Computer Science*, vol. Volume 4 - Proceedings of MFPS XL, Dec. 11, 2024, ISSN: 2969-2431. DOI: 10.46298/entics.14638. [Online]. Available: <https://entics.episciences.org/paper/view/id/14638> (visited on 12/12/2024).
- [7] E. Di Lavore, A. Gianola, M. Román, N. Sabadini, and P. Sobociński, “A canonical algebra of open transition systems”, in *International Conference on Formal Aspects of Component Software*, Springer, 2021, pp. 63–81.
- [8] J. L. Fiadeiro and V. Schmitt, “Structured co-spans: An algebra of interaction protocols”, in *International Conference on Algebra and Coalgebra in Computer Science*, Springer, 2007, pp. 194–208.
- [9] B. Fong, “Decorated cospans”, *Theory & Applications of Categories*, vol. 30, 2015.

References III

- [10] A. Gianola, S. Kasangian, and N. Sabadini, “Cospan/span(graph): An algebra for open, reconfigurable automata networks”, in *7th Conference on Algebra and Coalgebra in Computer Science (CALCO 2017)*, Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2017.
- [11] M. Grandis, *Higher Dimensional Categories: From Double to Multiple Categories*. World Scientific, 2019.
- [12] R. Haugseng, F. Hebestreit, S. Linskens, and J. Nuiten, “Two-variable fibrations, factorisation systems and ∞ -categories of spans”, arXiv: 2011.11042 [math]. (Sep. 20, 2023), [Online]. Available: <http://arxiv.org/abs/2011.11042> (visited on 03/15/2024), pre-published.
- [13] B. Jacobs, *Introduction to Coalgebra*. Cambridge University Press, 2017, vol. 59. [Online]. Available: <https://books.google.com/books?hl=en&lr=&id=tApQDQAAQBAJ&oi=fnd&pg=PR7&dq=jacobs+introduction+to+coalgebra&ots=Zm3dpD7Sbo&sig=R5uGNh6AARnrbrRu7r04006t7As> (visited on 10/30/2023).

References IV

- [14] A. Joyal, M. Nielsen, and G. Winskel, “Bisimulation from open maps”, *Information and Computation*, vol. 127, no. 2, pp. 164–185, 1996. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/S0890540196900577> (visited on 04/25/2024).
- [15] P. Karazeris and J. Velebil, “Representability relative to a doctrine”, *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, vol. 50, no. 1, pp. 3–22, 1 2009. [Online]. Available: http://www.numdam.org/item/CTGDC_2009__50_1_3_0/.
- [16] P. Katis, N. Sabadini, and R. F. Walters, “Bicategories of processes”, *Journal of Pure and Applied Algebra*, vol. 115, no. 2, pp. 141–178, 1997.
- [17] P. Katis, N. Sabadini, and R. F. Walters, “Span(graph): A categorical algebra of transition systems”, in *International Conference on Algebraic Methodology and Software Technology*, Springer, 1997, pp. 307–321.

References V

- [18] P. Katis, N. Sabadini, and R. F. Walters, “Feedback, trace and fixed-point semantics”, *RAIRO-Theoretical Informatics and Applications*, vol. 36, no. 2, pp. 181–194, 2002.
- [19] C. Kupke and Y. Venema, “Coalgebraic Automata Theory: Basic Results”, *Logical Methods in Computer Science*, vol. Volume 4, Issue 4, p. 1203, Nov. 21, 2008, ISSN: 1860-5974. DOI: 10.2168/LMCS-4(4:10)2008. [Online]. Available: <https://lmcs.episciences.org/1203> (visited on 10/30/2023).
- [20] S. Libkind, A. Baas, E. Patterson, and J. Fairbanks, “Operadic modeling of dynamical systems: Mathematics and computation”, 2021, Available at <https://arxiv.org/abs/2105.12282>.
- [21] J. Master, “The Open Algebraic Path Problem”, version 2. arXiv: 2005.06682 [math]. (Jan. 10, 2021), [Online]. Available: <http://arxiv.org/abs/2005.06682> (visited on 01/29/2025), pre-published.
- [22] D. J. Myers, “Double categories of open dynamical systems”, 2020. [Online]. Available: <https://arxiv.org/abs/2005.05956>.

References VI

- [23] D. J. Myers, *Categorical Systems Theory*. 2021, (Work in progress). [Online]. Available: <http://davidjaz.com/Papers/DynamicalBook.pdf> (visited on 05/14/2022).
- [24] N. Niu and D. I. Spivak, “Polynomial Functors: A Mathematical Theory of Interaction”, arXiv: 2312.00990 [math]. (Dec. 1, 2023), [Online]. Available: <http://arxiv.org/abs/2312.00990> (visited on 05/21/2024), pre-published.
- [25] J. J. Rutten, “Universal coalgebra: A theory of systems”, *Theoretical computer science*, vol. 249, no. 1, pp. 3–80, 2000. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/S0304397500000566> (visited on 10/30/2023).
- [26] P. Schultz and D. I. Spivak, *Temporal Type Theory: A Topos-Theoretic Approach to Systems and Behavior* (Progress in Computer Science and Applied Logic), 1st ed. Birkhäuser Cham, Feb. 11, 2019, VIII, 235, ISBN: 978-3-030-00703-4. [Online]. Available: <https://doi.org/10.1007/978-3-030-00704-1>.

References VII

- [27] B. Shapiro and D. I. Spivak, “Dynamic categories, dynamic operads: From deep learning to prediction markets” ,, 2022, Available at <https://arxiv.org/abs/2205.03906>.
- [28] D. I. Spivak, “The operad of wiring diagrams: Formalizing a graphical language for databases, recursion, and plug-and-play circuits” ,, 2013, Available at <https://arxiv.org/abs/1305.0297>.
- [29] D. I. Spivak, “Generalized Lens Categories via functors $\{F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}\}$ ”, arXiv: 1908.02202. (Mar. 17, 2022), [Online]. Available: <http://arxiv.org/abs/1908.02202> (visited on 10/23/2024), pre-published.
- [30] D. Vagner, D. I. Spivak, and E. Lerman, “Algebras of open dynamical systems on the operad of wiring diagrams” ,, 2014, Available at <https://arxiv.org/abs/1408.1598>.