

# Contextads abridged

based on joint work with David Jaz Myers ([arxiv:2410.21889](https://arxiv.org/abs/2410.21889))

**Matteo Capucci**

University of Strathclyde / Independent Researcher (ARIA Creator)

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## Why

1. Cokleisli arrows for a comonad  $(D, \varepsilon, \delta)$ :

$$\frac{DA \xrightarrow{f} B \quad DB \xrightarrow{g} C}{DA \xrightarrow{\delta} DDA \xrightarrow{Df} Db \xrightarrow{g} C}$$

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2. Parametric arrows for the action  $(C, \triangleleft)$  of a monoidal category  $\mathcal{M}$ :

$$\frac{\left( P \in \mathcal{M}, A \triangleleft P \xrightarrow{f} B \right) \quad \left( Q \in \mathcal{M}, B \triangleleft Q \xrightarrow{g} C \right)}{\left( P \otimes Q \in \mathcal{M}, A \triangleleft (P \otimes Q) \xrightarrow{\tilde{\delta}} (A \triangleleft P) \triangleleft Q \xrightarrow{f \triangleleft Q} B \triangleleft Q \xrightarrow{g} C \right)}$$

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### 3. Dependently parameterized functions:

$$\frac{\left( P : A \rightarrow \text{Type}, (a : A) \times P(a) \xrightarrow{f} B \right) \quad \left( Q : B \rightarrow \text{Type}, (b : B) \times Q(b) \xrightarrow{g} C \right)}{\left( P \times f^*Q : A \rightarrow \text{Type}, (a : A) \times ((p : P(a)) \times Q(f(a, p))) \xrightarrow{\sim} ((a : A) \times (p : P(a))) \times Q(f(a, p)) \rightarrow (b : B) \times Q(b) \xrightarrow{g} C \right)}$$

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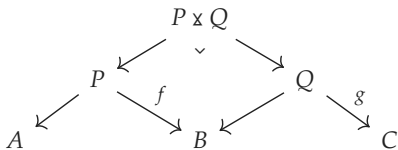
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or equivalently



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- ▶ M. Capucci and D. J. Myers, “Contextads as Wreaths; Kleisli, Para, and Span Constructions as Wreath Products”,, DOI: [10.48550/arXiv.2410.21889](https://doi.org/10.48550/arXiv.2410.21889)
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**Today:** just the basics.

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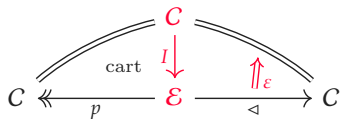
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thus a left-fibrant span

$$C \xleftarrow{p} \mathcal{E} \xrightarrow{\triangleleft} C$$

together with

3. a **trivial extension** map and a **counitor**

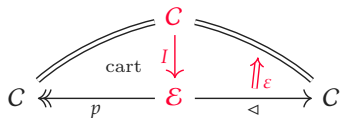


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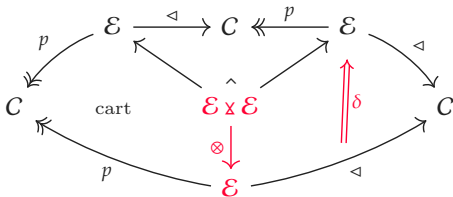
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4. an **extension combination** map and a **coassociator**:



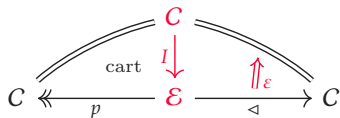
$$\begin{pmatrix} P \\ A \end{pmatrix} \otimes \begin{pmatrix} Q \\ A \triangleleft P \end{pmatrix} \in \mathcal{E}_A$$

$$\delta : A \triangleleft (P \otimes Q) \rightarrow (A \triangleleft P) \triangleleft Q$$

+ **left and right unitors**, and **associators** defined as certain  $p$ -vertical maps + **coherences**

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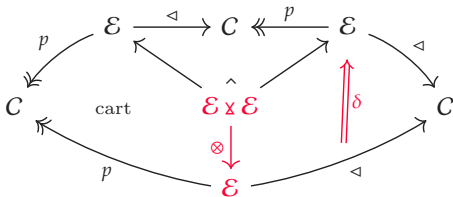
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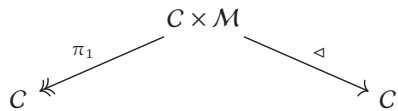
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#### Definition

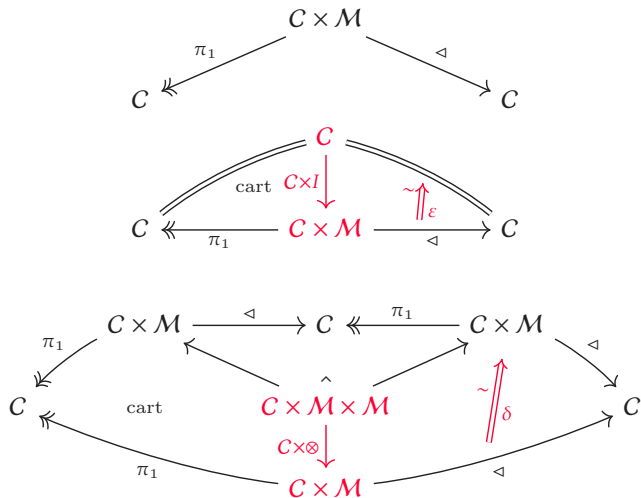
A **discrete contextad** is a contextad whose left leg is a discrete fibration.

These are given by a carrier  $C \xleftarrow{p} \mathcal{E} \xrightarrow{\triangleleft} C + \varepsilon, \delta +$  triangle and pentagon.

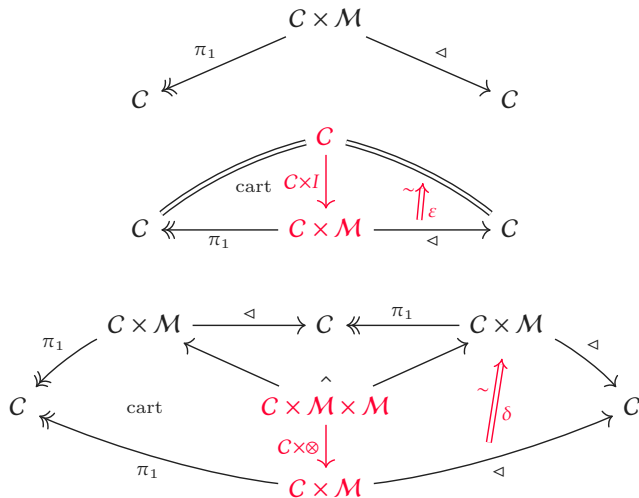
Examples:  $\mathcal{M}$ -actegory  $(\mathcal{C}, \triangleleft)$



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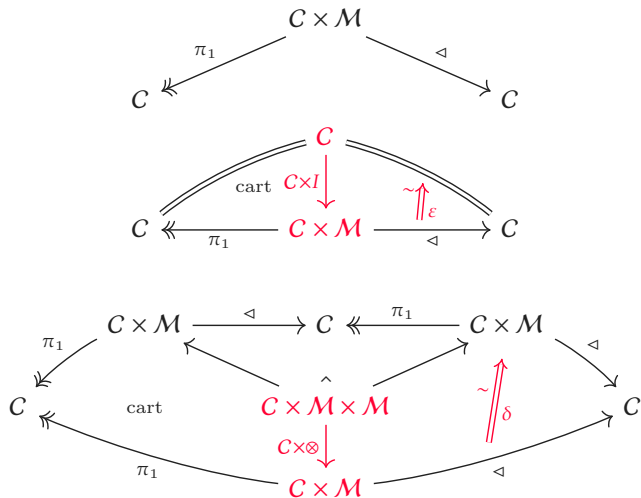


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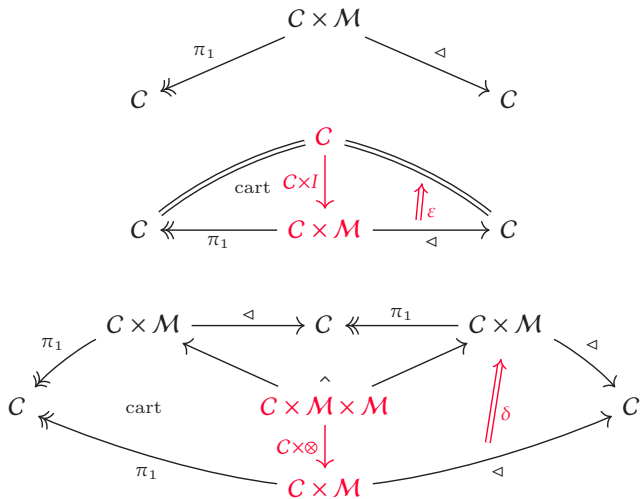
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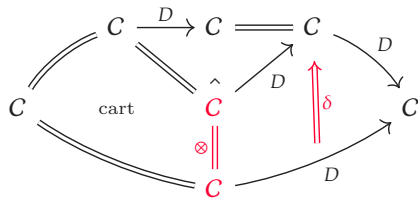
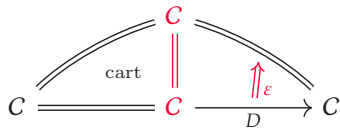
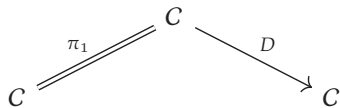
Unitors and associators and coherences are borrowed from  $\mathcal{M} \rightsquigarrow \text{discrete}$  when  $\mathcal{M}$  is (e.g. grading with a monoid).

Examples: **colax**  $\mathcal{M}$ -actegory  $(C, \triangleleft)$  aka **graded comonad**

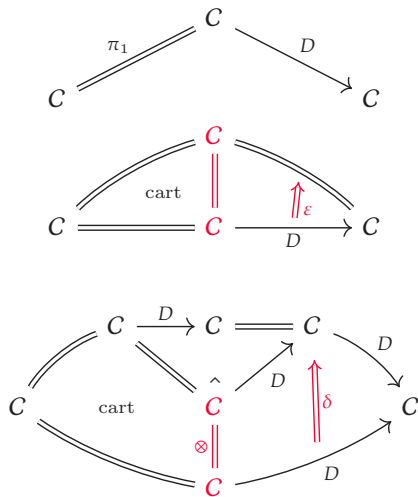


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Examples: **colax** 1-actegory  $(C, D)$  aka **comonad**

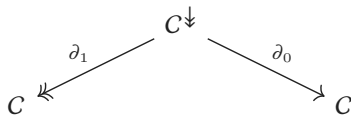


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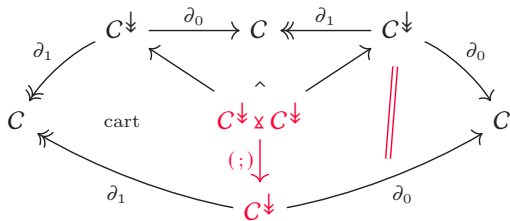
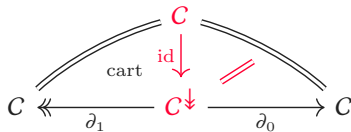
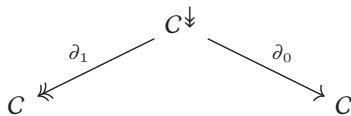


Always discrete!

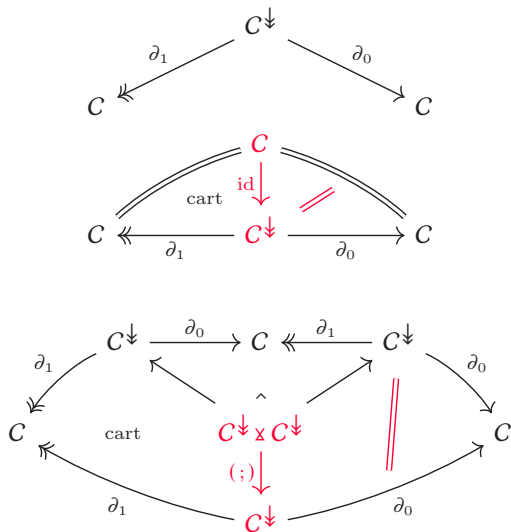
Examples:  $C$  equipped with strong display maps ( $\rightarrow$ )



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Comprehension categories work similarly, but yield in general a non-strict contextad.

## (Kinda New) Example: PRA monads as “dependently graded comonads”

Let  $\mathcal{C}$  have a terminal object.

### Definition

A **PRA monad** on  $\mathcal{C}$  is a monad  $T : \mathcal{C} \rightarrow \mathcal{C}$  such that  $T$  has a parametric left adjoint:

$$\mathcal{C} \xrightleftharpoons[T!]{L} \mathcal{C}/T1 \xrightarrow{\partial_0} \mathcal{C}$$

e.g. when  $\mathcal{C} = \mathbf{Set}$  and  $T = \sum_{s \in S} y^{P(s)}$  polynomial,

$$L(X \xrightarrow{k} S) = (x : X) \times P(k(x)).$$

In fact, we get a discrete contextad, the “dependently graded comonad” *transposing*  $T$ :

$$\begin{array}{ccc} & \mathcal{C}/T1 & \\ \partial_0 \swarrow & & \searrow L \\ \mathcal{C} & & \mathcal{C} \end{array}$$

## Ctx construction

**Quick definition.** Given a contextad  $C \xleftarrow{p} \mathcal{E} \xrightarrow{\triangleleft} C$ , the double category  $\mathbf{Ctx}(\triangleleft)$  is so comprised:

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2. Its loose 1-cells  $A \dashrightarrow B$  are **contextful arrows**, thus pairs of a context  $P \in \mathcal{E}_A$  and a 1-cell  $f : A \triangleleft P \rightarrow B$  in  $C$ . Units are the pairs  $(I, \varepsilon)$ , and composition is so defined:

$$\frac{\left( P \in \mathcal{E}_A, A \triangleleft P \xrightarrow{f} B \right) \quad \left( Q \in \mathcal{E}_B, B \triangleleft Q \xrightarrow{g} C \right)}{\left( P \otimes f^*Q \in \mathcal{E}_A, A \triangleleft (P \otimes f^*Q) \xrightarrow{\delta} (A \triangleleft P) \triangleleft f^*Q \xrightarrow{f \triangleleft Q} B \triangleleft Q \xrightarrow{g} C \right)}$$

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3. Its squares are arrangements as below left where  $\varphi : P \rightarrow P'$  is a map in  $\mathcal{E}$  over  $h$  making the diagram below right commute.

$$\begin{array}{ccc} A & \xrightarrow{(P,f)} & B \\ h \downarrow & \varphi \Downarrow & \downarrow k \\ A' & \xrightarrow{(P',f')} & B' \end{array}$$

$$\begin{array}{ccc} A \triangleleft P & \xrightarrow{f} & B \\ h \triangleleft \varphi \downarrow & & \downarrow k \\ A' \triangleleft P' & \xrightarrow{f'} & B' \end{array}$$

## Examples

1. (colax) actions of monoidal categories  $(C, \triangleleft)$  ((Hermida and Tennent 2012; Capucci, Gavranović, Hedges, and Rischel 2022)):

$$\mathbf{Ctx}(\triangleleft) = \mathbf{Para}(\triangleleft)$$

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3. display map cats  $(C, \twoheadrightarrow)$

$$\mathbf{Ctx}(\partial_0) = \mathbf{Span}(C, \twoheadrightarrow)$$

## Idea: transposing monads to dependently graded comonads

**Theorem.** If  $T$  is a PRA monad and  $C \xleftarrow{\partial_0} C/T1 \xrightarrow{L} C$  its associated contextad, then there is an identity-on-objects isomorphism  $\mathbf{Kl}(T) \simeq \mathbf{Ctx}(L)$  given on arrows by the canonical isomorphism:

$$X \xrightarrow{f} TY \quad \Leftrightarrow \quad \begin{cases} k : X \xrightarrow{(T!)f} T1 \\ \hat{f} : Lk \rightarrow Y \end{cases}$$

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This allows to see polynomial effects as “contextful effects” (Kieburtz 1999; Uustalu and Vene 2008) realized by a **(dependently graded) comonad**.

**Thanks!**

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