Syntax and Semantics of QPL

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Why

Definition. Let $f : X \to [0, \infty]$ be an integrable function over X measure space. Its **softmax** is the probability distribution $X \to [0, 1]$ defined as

$$(\operatorname{softmax} f)(x^*) = \frac{f(x^*)}{\int_{x \in X} f(x)\, dx}.$$

Definition. Let $f : X \to \mathbb{R}$ be a function. Its **argmax** is the predicate $X \to \mathbf{Prop}$ defined as

$$(\operatorname{argmax} f)(x^*) = \forall x \in X,\ f(x) \leq f(x^*).$$

Why

Section 1.2. Quantitative Alethic Modal Logic > **The semantics** [unstable/639] [edit]

Let $[\![-]\!] : \mathsf{LT} \to P(W)$ be a model of the propositional fragment in predicates over a set of **possible worlds** W. Fix now a Kripke model (W, R), where $R : W \nrightarrow W$ is a relation called **accessibility**. Then the modalities are interpreted as follows (see Simpson p. 50):

$$[\![\Diamond\varphi]\!](w) = \exists v \in W.\, wRv \wedge \varphi(v),$$

and dually

$$[\![\Box\varphi]\!](w) = \forall v \in W.\, wRv \to \varphi(v).$$

Let $[\![-]\!] : \mathsf{LT} \to L(W)$ be a model of the propositional fragment in quantitative predicates over a probability space of **possible worlds** W. Fix now a Kripke model (W, R), where $R : W \times W \to [0, \infty]_\otimes$ is a quantitative relation called **accessibility**. Then the modalities are interpreted as follows:

$$[\![\Diamond^p\varphi]\!](w) = [\![\exists^p v \in W.\, wRv \otimes \varphi(v)]\!]_\otimes,$$

and dually

$$[\![\Box^p\varphi]\!](w) = [\![\forall^p v \in W.\, wRv \multimap \varphi(v)]\!]_\otimes.$$

For instance, let R be a Markov kernel, i.e. $wRv = p_w(v)$ for some probability density p_w. Then $\Diamond^1\varphi$ has then a pretty straightforward interpretation: it is the *probability φ will be true in the next state*:

$$[\![\Diamond^1\varphi]\!](w) = [\![\exists^1 v \in W.\, wRv \otimes \varphi(v)]\!]_\otimes(w) = \int_{v\in W} p_w(v)\varphi(v)\ \mathrm{d}\,v.$$

Why

In general, the fact that $\int$ 'behaves like' '$\exists$/*colim*' has been noted before:

From (Loregian 2021):

$$\int_{x\in\mathbb{R}} f(x) \circ \delta(x-y)dx = f(y)$$

$$\int^{X} FX \times \mathcal{C}(Y,X) \cong FY$$

The analogy between the pairing of a function and a delta distribution, and the ninja Yoneda lemma.

Why

Generally speaking, a ‘soft logic’ which can be used to deal with approximate/probabilistic/quantititative notions of evidence/truth can be used for pretty much anything:

1. probabilistic programming & checking
2. NN verification
3. approximate reasoning
4. interpretability
5. probability theory (the ‘original’ QPL)

How

1. Use the reals as truth values → $[0,1]$, $\mathbb{Q}$, ...
2. Bootstrap syntax & semantics *together*
3. Work parametric on ‘hardness’ and recover classical reasoning in the limit

What

1. A propositional BI-like logic extending Polynomial Lawvere logic of Bacci et al.
2. A ‘structured sequent calculus’ graded by hardness
3. A categorical semantics in locally graded enriched posets of measurable functions

The reals

$$
\begin{tikzcd}[column sep=huge]
 & \textbf{multiplicative} \arrow[r, leftrightarrow, "\textbf{Napierian duality}"] & \textbf{additive} \\
\textbf{negative} \arrow[dd, leftrightarrow, "\textbf{de Morgan duality}"'] & {[0,1]_{\otimes}} \arrow[d, hook] \arrow[r, bend right=10, "-\log"'] \arrow[dd, dotted, leftrightarrow, bend right=40, "1/-"'] & {[-\infty,0]^{\mathrm{op}}_{\oplus}} \arrow[l, bend right=10, "1/\exp"'] \arrow[l, phantom, "\sim"] \arrow[d, hook] \arrow[dd, dotted, leftrightarrow, bend left=40, "-(-)"] \\
 & {[0,\infty]_{\otimes}} \arrow[r, bend right=10, "-\log"'] & {[-\infty,\infty]^{\mathrm{op}}_{\oplus}} \arrow[l, bend right=10, "1/\exp"'] \arrow[l, phantom, "\sim"] \\
\textbf{positive} & {[1,\infty]_{\otimes}} \arrow[u, hook] \arrow[r, bend right=10, "-\log"'] & {[0,\infty]^{\mathrm{op}}_{\oplus}} \arrow[l, bend right=10, "1/\exp"'] \arrow[l, phantom, "\sim"] \arrow[u, hook]
\end{tikzcd}
$$

Bacci et al.

The multiplicative reals

$a \otimes b$	0	$a \in (0,\infty)$	∞
0	0	0	0
$b \in (0,\infty)$	0	ab	∞
∞	0	∞	∞

a^*	0	$a \in (0,\infty)$	∞
	∞	$1/a$	0

$a \otimes^* b$	0	$a \in (0,\infty)$	∞
0	0	0	∞
$b \in (0,\infty)$	0	ab	∞
∞	∞	∞	∞

$a \multimap b$	0	$a \in (0,\infty)$	∞
0	∞	0	0
$b \in (0,\infty)$	∞	b/a	0
∞	∞	∞	∞

$*$-aut quantale

$([0,\infty], \leq)$

The multiplicative reals

$a \oplus^p b \xrightarrow{p \to \infty} a \vee b$

add q   add l

$p \in (0, \infty)$

$a \oplus^p b$	0	$a \in (0,\infty)$	∞
0	0	a	∞
$b \in (0,\infty)$	b	$(a^p + b^p)^{1/p}$	∞
∞	∞	∞	∞

$a \oplus^{-p} b$	0	$a \in (0,\infty)$	∞
0	0	0	0
$b \in (0,\infty)$	0	$\dfrac{1}{(1/a^p + 1/b^p)^{1/p}}$	b
∞	0	a	∞

p-sums   harmonic p-sum

$$\wedge \quad \cdots \quad \oplus^{-p} \quad \cdots \quad \oplus^{*} \quad \cdots \quad \infty \quad \cdots \quad \oplus \quad \cdots \quad \oplus^{p} \quad \cdots \quad \vee$$

$$-\infty \longleftarrow\!\!-\!\!-\!\!-\!\!-\; -1 \;-\!\!-\!\!-\; 0 \;-\!\!-\!\!-\; 1 \;-\!\!-\!\!-\!\!-\!\!\longrightarrow +\infty$$

The multiplicative reals

polarity		additive			multiplicative
duality $a^* := 1/a$	**positive**	**false** $:= 0$ $a \vee b$	$\xleftarrow{p\to\infty}$	$\mathbf{0} := 0$ $a \oplus^p b$	$\mathbf{1} := 1$ $a \otimes b$
	negative	**true** $:= \infty$ $a \wedge b$	$\xleftarrow{p\to\infty}$	$\top := \infty$ $a \oplus^{-p} b$	$\bot := 1$ $a \otimes^* b$

lax commutativity	$1 = 1$	$\underline{a \otimes^* (b \otimes c) \leq (a \otimes^* b) \otimes c}$
distributivity	$a \otimes 0 = 0$	$a \otimes (b \oplus^p c) = (a \otimes b) \oplus^p (a \otimes c)$
graded duoidality	$0 \leq 1$	$(a \otimes b) \oplus^{p \oplus^* q} (c \otimes d) \leq (a \oplus^p c) \otimes (b \oplus^q d)$

propositional

QPL

? calculus (sequent)

? algebraic semantics

↳ Hölder's ineq

$p = q = \infty$ $\quad \oplus^\infty = \vee$

The spectrum of p-means

p-norms

≥ 0

Definition. For any $p \in (-\infty, \infty)$, $p \neq 0$, the ***p*-mean** of a finite set of numbers $(a_i)_{i \in I}$ is

$$\int^p a_i \, di = \left(\int a_i^p \, di \right)^{1/p}$$

$$\int_{i \in I}^{p} a_i := \left(\bigoplus_{i \in I} \frac{a_i^p}{|I|} \right)^{1/p}$$

$I = \{1, \ldots, n\}$

$di = \frac{1}{|I|} = \frac{1}{n}$

when $p < 0$ we call p-means **harmonic**.

One extends the above definition to $p = \pm\infty$ by taking suitable limits.

$$\int^{-0} \leq \int^{0}$$

$$\begin{array}{ccccccccccccc}
\bigwedge & \cdots & \int^{-p} & \cdots & \int^{*} & & & \int & \cdots & \int^{p} & \cdots & \bigvee \\
-\infty & \text{———} & -p & \text{———} & -1 & \text{——} & 0 & \text{——} & 1 & \text{———} & p & \longrightarrow & \infty
\end{array}$$

"$\wedge$" ess inf

"$\vee$" ess suprema

The amazing p-means

$\forall i.\ a \multimap \psi(i) = a \multimap \exists i\, \psi(i)$

$\exists i \sim \int^{p}$

$\forall i \sim \int^{-p}$

Lemma. Harmonic sum

1. satisfies the identity:

$$\boxed{\int_{i\in I}^{-p} \frac{a}{\psi(i)} = \frac{a}{\int_{i\in I}^{p} \psi(i)}}$$

2. satisfies the Fubini property:

$$\int_{i\in I}^{-p} \int_{j\in J}^{-p} \varphi(i,j) = \int_{(i,j)\in I\times J}^{-p} \varphi(i,j) = \int_{j\in J}^{-p} \int_{i\in I}^{-p} \varphi(i,j),$$

3. is homogeneous (i.e. multiplication distributes over it):

$$k \otimes \int_{i\in I}^{-p} \varphi(i) = \int_{i\in I}^{-p} k \otimes \varphi(i),$$

Lemma. Harmonic sum

1. is monotonic in the argument: if, for each $i \in I$, $\varphi(i) \leq \psi(i)$, then

$$\int_{i\in I}^{-p} \varphi(i) \leq \int_{i\in I}^{-p} \psi(i),$$

2. is antitonic in the index: when $J \subseteq I$, one has

$$\int_{i\in I}^{-p} \varphi(i) \leq \int_{j\in J}^{-p} \varphi(j).$$

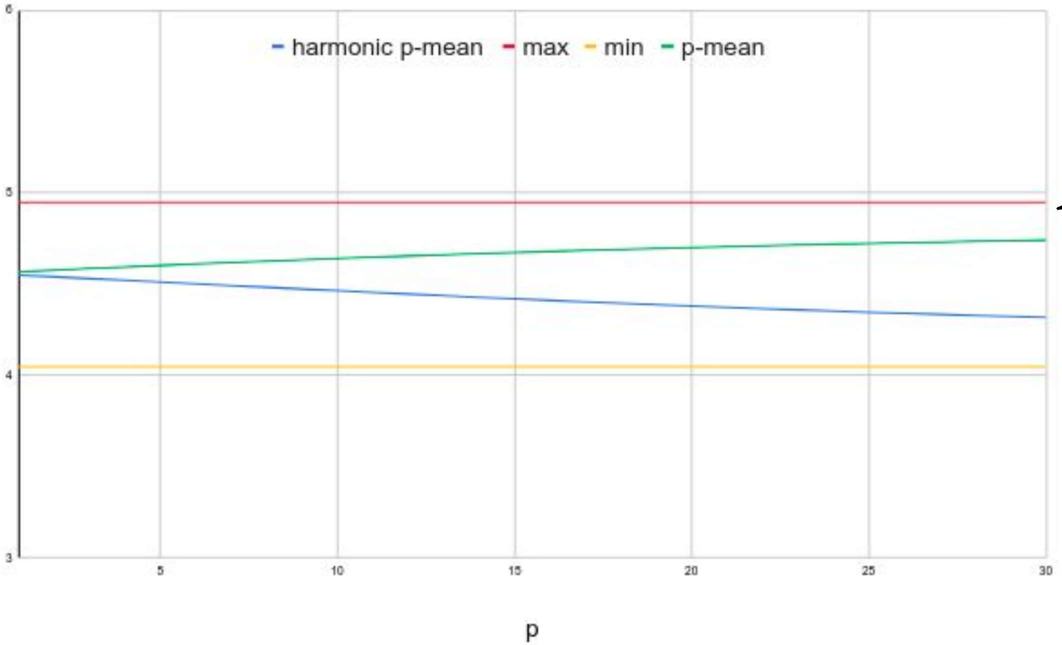
The spectrum of p-means

We have $\bigwedge \leq \int^{-p} \leq \int^{p} \leq \bigvee$, with the (exterior) gap narrowing as p increases:

$p \leq q$

$\int^{-q} \leq \int^{-p}$

$\int^{p} \leq \int^{q}$



$\uparrow p \to -\infty$

$\downarrow p \to \infty$

Unlike p-sums, p-means **compensate for cumulative effects**.

Means as bounded quantifiers

Idea. Mean and harmonic mean correspond to **bounded quantification**:

$$\exists i.\, (i \in I) \wedge a(i) \quad \rightsquigarrow \quad \bigoplus_{i \in I}^{p} \frac{1}{|I|} \otimes a(i)$$

A mean is a just an integral over a probability space so we can directly generalize:

$$\boxed{\exists i \in I.\, a(i) \quad \rightsquigarrow \quad \int_{i \in I}^{p} a(i)\, di.}$$

for I any probability space. Duality also suggests this interpretation since

$$\forall i \in I.\, a(i) = \neg \exists i \in I.\, \neg a(i) \quad \rightsquigarrow \quad \int_{i \in I}^{-p} a(i) = \left(\int_{i \in I}^{p} a(i)^{*} di \right)^{*},$$

i.e. we dualize $a(i)$ but not the domain of quantification.

QPL: intended semantics

We'd like quantifiers to be integrals, hence we'd like predicates to be measurable functions.

$\varphi \vdash_I \psi$??

predicates : $L(I, \mathrm{d}i) = \{\varphi : I \to [0, \infty]_{\otimes} \text{ measurable}\}$

↳ prob spaces "contexts" / domains

We aim to define an hyperdoctrine so that p-means are suitable adjoints…

$$L(I, di) = (L(I), \leq) \qquad \left(\begin{array}{c}\text{like } [0,\infty]_{\otimes} \\ a \leq b\end{array}\right)$$

$$\hookrightarrow \varphi \leq \psi \iff \forall i \;\; \varphi(i) \leq \psi(i)$$

"hard"

$$\varphi \vdash_\infty \psi$$

e.g. $\;? \;\varphi \vdash_\infty \psi(x) = \varphi^{(x,y)} \vdash_\infty \psi(x) \qquad \pi_X : X \times Y \to X$

ptwise order

$$\bigvee_{y \in Y} \varphi(x,y)$$

instead $\varphi, \psi : I \to [0,\infty]$

$$\varphi \vdash_p \psi = \int_{i \in I}^{-p} \varphi(i) \multimap \psi(i) \, di \in [0,\infty]!$$

$\varphi : X \times Y \to [0,\infty]$, $\psi : X \to [0,\infty]$

$$\left(\vdash_p : L(I) \times L(I) \to [0,\infty]\right)_{p \in [0,\infty]}$$

$$\exists_y^p \varphi(x,y) \vdash_p \psi(x) = \varphi(x,y) \vdash_p \underset{\psi(\pi_X(x,y))}{\overset{}{\psi(x)}}$$

$$\int_x^{-p} \left(\int_y^p \varphi(x,y)\right) \multimap \psi(x) \, dx = \int_{x,y}^{-p} \left(\varphi(x,y) \multimap \psi(x)\right) dx$$

$$\| \qquad \qquad \qquad \qquad \qquad \|$$

|| "fund prop of harmonic p-means" || Fubini

$$\int_x^{-p} \frac{\psi(x)}{\int_y^p \varphi(x,y)\,dy}\,dx = \underbrace{\int_x^{-p}\int_y^{-p}}\frac{\psi(x)}{\varphi(x,y)}\,dy\,dx$$

$$\text{id/ax}\ \frac{}{\varphi \vdash_p \varphi}$$

$$\cancel{\frac{\varphi \vdash_p \psi \quad \psi \vdash_p \chi}{\varphi \vdash_p \chi}}\ \text{trans/cut}$$

On Quant for Quant Reasoning — M.C.

IDEA Hölder's inequality

$$\frac{\varphi \vdash_p \psi \qquad \psi \vdash_q \chi}{\varphi \vdash_{p \circledast q} \chi}$$

$$\|fg\|_{p\circledast q} \leq \|f\|_p \|g\|_q \qquad p,q \in (0,\infty]$$

$$1 = \frac{1}{p} + \frac{1}{q} \qquad p,q \geq 1$$

$L : \mathrm{Prob}^{op} \longrightarrow \mathcal{K} \rightsquigarrow$ "algebras of preds"

???

~~v2.0~~

$\mathcal{K}$ is the 2-cat of locally $\mathcal{M}$-graded $\mathcal{V}$-enriched categories

$\mathcal{M} = ([0,\infty], \oplus^*, \infty) \qquad \mathcal{V} = [0,\infty]_{\otimes}$

$$\mathcal{K} \ni \left(L(I), \left(\vdash_p : L(I) \times L(I) \longrightarrow [0,\infty]_{\otimes}\right)_{p \in [0,\infty]_{\oplus^*}}\right)$$

(graded refl) $\quad 1 \leq (\varphi \vdash_\infty \varphi)$ trivial!

(graded trans) $\quad (\varphi \vdash_p \psi) \otimes (\psi \vdash_q \chi) \leq (\varphi \vdash_{p \oplus^* q} \chi)$

Hölder's inequality

TH

$L : \mathrm{Prob}^{op} \longrightarrow \mathcal{K}$ hyperdoctrine

" ~~primary~~ (prods + $L(\pi_X)$ has both adjoints*)

* $\exists^\infty \dashv \pi_X^* \dashv \forall^\infty$ ✓

p-adjoints $f \dashv_p g$ in $\mathcal{K}$ $(f(x) \le_p y) = (x \le_p g(y))$

$A \to B$ $\forall x, y \in A, B$ locally r-graded V-enriched cats

$L(\pi_X)$ has both p-adjs for all $p \in (0, \infty]$

$PX = \Delta X$

$(X, \mu) \times LX \xrightarrow{\epsilon} [0, \infty]$

$(X, f) \overset{?}{\mapsto} \int f \, d\mu$

$$\int^p \dashv_p \pi_X^* \dashv \int^{-p}$$

/qpl-halfway.xml

$$\exists^p \dashv \pi_X \dashv \forall^p$$