

What if additives were actually addition?

Matteo Capucci

joint with Bob Atkey, Katya Komendantskaya, Charles Grellois

Motivation

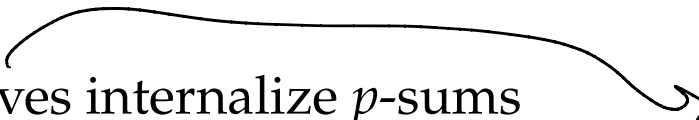
1. Correct-by-training verification of gradient-based ML models
2. 'Semiotics' of quantitative methods

Motivation

	Fuzzy	Differentiable
Nice proof theory/ model theory	✓	✗
Nice semantics	✗	✓

p QLL

' p -soft Quantitative Linear Logic'

- ' p -hard', (soft): additives internalize p -sums  \times , $\dot{\times}$
- Quantitative in the sense of Lawvere, *i.e.* enriched in the reals
- Sequent calculus *qualitatively* identical to **MALL**
Conservative and adequate over MALL
- Varying $0 < p \leq \infty$, it bridges *soft* ($p < \infty$) and *hard* ($p = \infty$)

Results

- Cut-elimination, soundness and completeness, approximation

Positive reals: multiplicative structure

$[0, \infty]$

(conjunctive) multiplication, i.e. the following extension of the usual arithmetic multiplication:

$a \otimes b$	0	$a \in (0, \infty)$	∞
0	0	0	0
$b \in (0, \infty)$	0	ab	∞
∞	0	∞	∞

$$0 \otimes \infty = 0$$

Crucially, **inversion** yields a duality $(-)^* : [0, \infty]^{\text{op}} \rightarrow [0, \infty]$

a^*	0	$a \in (0, \infty)$	∞
∞	∞	$1/a$	0

which is also uniquely characterized by the property

$$(1) \quad a \otimes b \leq c^* \iff a \leq (b \otimes c)^* \quad (3)$$

$$\curvearrowright a \otimes^* b = (a^* \otimes b^*)^*$$

By de Morgan duality, we obtain a second ‘multiplicative’ operation, \otimes^* (**disjunctive multiplication**), which differs from the first only for the fact that $0 \otimes^* \infty = \infty$. In particular, \otimes linearly distributes over \otimes^* in the sense of [20]:

$$(a \otimes^* b) \otimes c \leq a \otimes^* (b \otimes c). \quad (4)$$

The residuation is explicitly given by $a \multimap b = a^* \otimes^* b$. This means $a \leq b \multimap c \iff a \otimes b \leq c$.

$a \multimap b$	0	$a \in (0, \infty)$	∞
0	∞	0	0
$b \in (0, \infty)$	∞	b/a	0
∞	∞	∞	∞

$$0 \otimes^* \infty = \infty \quad \text{vs} \quad 0 \otimes \infty = 0$$

Positive reals: additive structure

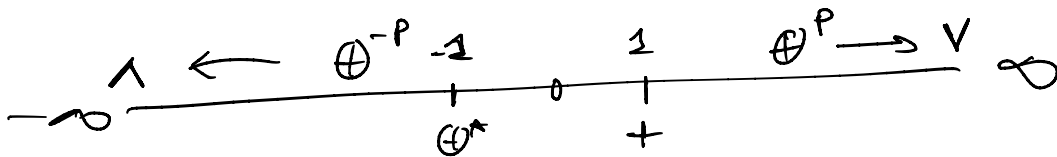
Now, on the poset $[0, \infty]$, consider **sum** \oplus , trivially extended to ∞ , and its de Morgan dual, **harmonic sum** ([29]):

$$a \oplus^* b := (a^* \oplus b^*)^*. \quad (6)$$

$$a \oplus^* b = \frac{1}{\frac{1}{a} \oplus \frac{1}{b}}$$

Choosing $0 < p < \infty$, we can conjugate these operations by exponentiation⁴ to obtain **p -sum** and **harmonic p -sums**:

$$a \oplus^p b := (a^p \oplus b^p)^{1/p}, \quad a \oplus^{-p} b := (a^p \oplus^* b^p)^{1/p}.$$



Lemma 2.1 (Additive Collapse).

$$a \oplus^p b \xrightarrow{p \rightarrow \infty} a \vee b, \quad a \oplus^{-p} b \xrightarrow{p \rightarrow \infty} a \wedge b, \quad (7)$$

Positive reals

$$([0, \infty], \leq)$$

$$a \otimes (b \oplus^p c) = (a \otimes b) \oplus^p (a \otimes c)$$

← dist

additive

multiplicative

duality $a^* := 1/a$	$\rightarrow \oplus$	$(0, a \vee b) \xleftarrow{p \rightarrow \infty} (0, a \oplus^p b)$	$(1, a \otimes b)$
	$\rightarrow \ominus$	$(\infty, a \wedge b) \xleftarrow{p \rightarrow \infty} (\infty, a \oplus^{-p} b)$	dist $(1, a \otimes^* b)$

Table 1: Algebraic structure on $[0, \infty]$.

$\overbrace{\hspace{10em}}$
 isomix ($1 = 1$)
 \star -auton poset

 p -veals (fixed p)

Qualitative modes

In the **additive mode**, we consider $a = 0$ ‘false’ and everything else ‘true’. Thus the map casting from ‘quantitative’ to ‘qualitative’ truth is

$$0 < _ : [0, \infty] \longrightarrow \mathbf{Prop} \quad (15)$$

Notice this commutes with the conjunctive operations (\otimes and \oplus^{-p}), which are both mapped to \wedge . Likewise \oplus^p is mapped to \vee , but note \otimes^* is not preserved. Rather, we can only say that $a > 0$ and $b > 0$ imply $a \otimes^* b > 0$ (so \otimes^* laxly commutes with \wedge), and that $a \otimes^* b > 0$ implies $a > 0$ or $b > 0$ (\otimes^* colaxly commutes with \vee). Negation is not preserved either way.

In the **multiplicative mode**, we pick 1 as threshold of truth and consider

$$1 \leq _ : [0, \infty] \longrightarrow \mathbf{Prop} \quad (16)$$

Conjunctive operations (\otimes and \oplus^{-p}) laxly commute with \wedge , while disjunctive ones (\otimes^* and \oplus^p) colaxly commute with \vee . When $p = \infty$, \oplus^p and \oplus^{-p} are preserved strongly. We also have $a > 1$ implies $1 \leq a^*$.

Quantitative calculi

Sequent calculi work by ‘internalizing’ a fragment of the metatheory we call **alethic theory**

$$\frac{\Gamma \vdash \Delta, A \text{ and } \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \wedge_{\text{R}}$$

Introduction rules turn alethic-level operations on sequents to object-level operations.

Idea: change alethic theory to $[0, \infty]$ to restore the proof theoretic properties of the additives in a ‘fuzzy’ logic.

Quantitative calculi

In a quantitative calculus, sequents are combined in **structures**, which are entities defined as follows

$$\mathcal{H} ::= \Gamma \vdash \Delta \mid r \mid \mathcal{H} \otimes \mathcal{H} \mid \mathcal{H} \otimes^* \mathcal{H} \mid \mathcal{H} \oplus^p \mathcal{H} \mid \mathcal{H} \oplus^{-p} \mathcal{H} \quad (17)$$

$1, 0, \infty$

where $\Gamma \vdash \Delta$ ranges in the possible sequents in the language Φ while r ranges in $[0, \infty]$.

some \mathcal{K}
 $|| \oplus || = 2 \in [0, \infty]$

A structure of sequents involving no sequent is called **closed**, while a structure of sequents consisting of a single sequent is called **unary**. We remark that closed structures can be *evaluated* to real numbers, but *per se* are syntactic objects. We call the result of evaluating a closed structure \mathcal{K} its **alethic value** or **validity**, and write it as $|\mathcal{K}|$.

Quantitative calculi

A **rule** has the form $\frac{\mathcal{H}}{\mathcal{H}'}$. Its **premises** are the sequents appearing in \mathcal{H} and its **conclusions** those appearing in \mathcal{H}' . It is an **axiom** if the top structure of sequents is closed. A set of rules \mathcal{R} is a **calculus**. A **derivation** in the calculus \mathcal{R} is defined inductively as follows (where $\bullet \in \{\otimes, \otimes^*, \oplus^P, \oplus^{-P}\}$):

- (1) for each \mathcal{H} , the **identity rule** $\frac{\mathcal{H}}{\mathcal{H}}$ $\text{id}_{\mathcal{H}}$ is a derivation;
- (2) every rule in \mathcal{R} is a derivation,

(3) if $\begin{array}{c} \mathcal{H} \\ \vdots \\ D_1 \end{array}$ and $\begin{array}{c} \mathcal{I} \\ \vdots \\ D_2 \end{array}$ are derivations, so is $\begin{array}{c} \mathcal{H} \\ \vdots \\ D_1 \\ \vdots \\ D_2 \end{array}$

(4) if $\begin{array}{c} \mathcal{I} \\ \vdots \\ D_1 \end{array}$ and $\begin{array}{c} \mathcal{K} \\ \vdots \\ D_2 \end{array}$ are derivations, so is $\begin{array}{c} \mathcal{K} \\ \vdots \\ D_1 \bullet D_2 \end{array}$

We stress that a derivation can only mention finitely many rules.

$$\frac{\Gamma \vdash \Delta \quad \otimes \quad \Gamma' \vdash \Delta'}{\Gamma \vdash \Delta'' \quad \oplus \quad \infty}$$

Quantitative calculi

A **closed** derivation is a derivation where the top structure of sequents is closed. A **proof** is a closed derivation where the bottom structure of sequents is unary. The **validity** $|\pi|$ of a closed derivation π is the alethic value of its top structure of sequents.

Example 3.2. To exemplify the definition just given, here is a proof in $pQLL$, whose validity is computed by traversing the proof tree, obtaining $|1 \oplus^{-p} 1| = 2^{1/p}$:

$$2^{1/p} = \frac{\frac{1}{A \vdash A} \text{ AX} \quad \oplus^{-p} \quad \frac{1}{A \vdash A} \text{ AX}}{A \vee^p A \vdash A} \text{ V}_L^p \quad (19)$$

Quantitative calculi

The **provability** $|\Gamma \vdash \Delta|_{\mathcal{R}}$ of a sequent $\Gamma \vdash \Delta$ in the calculus \mathcal{R} is the supremum of the validity of its proofs:

$$|\Gamma \vdash \Delta|_{\mathcal{R}} = \bigvee \left\{ |\pi| \mid \begin{array}{c} \vdots \pi \text{ closed} \\ \Gamma \vdash \Delta \end{array} \right\} \quad (\text{provability})$$

By replacing $\Gamma \vdash \Delta$ with a structure of sequents, we extend the above definition⁹. Observe that, as consequence, any derivation from \mathcal{H} to \mathcal{H}' witnesses that $|\mathcal{H}| \leq |\mathcal{H}'|$.

Thus the mere existence of a proof does not qualify a sequent as provable, since a sequent might admit proofs of provability 0 (**invalid** proofs); likewise, when a given sequent does not admit any proof, its provability is $\sup \emptyset = 0$.

\rightarrow pQLL is "decidable"
i.e. this is a max

p QLL

fix $a \quad 0 < p \leq \infty$

Definition 3.5. Fix a set of **atomic propositions** V . A **formula of p QLL** with atoms in V is defined inductively as follows:

$$\begin{aligned} A & ::= a, a^* && \text{((negated) atomic prop. } a \in V) \\ & \quad | \underline{\mathbf{1}} | A \otimes A | A \otimes^* A && \text{(multiplicatives)} \\ & \quad | \underline{\perp} | \underline{\top} | A \vee^p A | A \wedge^p A && \text{(additives)} \end{aligned}$$

Negation $(-)^*$ is a syntactic transformation defined by

$$\begin{aligned} (a)^* & \equiv a^*, & (a^*)^* & \equiv a, \\ \mathbf{1}^* & \equiv \mathbf{1}, & \perp^* & \equiv \top, & \top^* & \equiv \perp, \\ (A \otimes B)^* & \equiv A^* \otimes^* B^*, & (A \otimes^* B)^* & \equiv A^* \otimes B^*, \\ (A \vee^p B)^* & \equiv A^* \wedge^p B^*, & (A \wedge^p B)^* & \equiv A^* \vee^p B^*, \end{aligned} \tag{25}$$

We also employ the abbreviation $A \multimap B \equiv A^* \otimes^* B$.

Cedents are (possibly) multisets of formulae, sequents are ordered pairs of cedents.

pQLL

(a) Structural Rules

$$\frac{1}{A \vdash A} \text{AX} \quad \frac{1}{\vdash} \text{EMP} \quad \frac{0}{\Gamma \vdash \Delta} \text{EFQ}$$

$$\frac{\Gamma \vdash A, \Delta \quad \otimes \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{CUT} \quad \frac{\Gamma \vdash \Delta \quad \otimes^* \quad \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{MIX}^*$$

(b) Multiplicative Rules

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \otimes_L \quad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \otimes^* B, \Delta} \otimes_R^*$$

$$\frac{\Gamma, A \vdash \Delta \quad \otimes \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \otimes^* B \vdash \Delta, \Delta'} \otimes_L^* \quad \frac{\Gamma \vdash A, \Delta \quad \otimes \quad \Gamma', \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'} \otimes_R$$

$$\frac{1}{1 \vdash} 1_L \quad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^* \vdash \Delta} {}^*_L \quad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^*, \Delta} {}^*_R \quad \frac{1}{\vdash 1} 1_R$$

See MALL

p QLL

(c) Additive Rules

$$\frac{\Gamma, A \vdash \Delta \quad \oplus^{-p} \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee^p B \vdash \Delta} \vee_L^p \quad \frac{\Gamma \vdash A, \Delta \quad \oplus^{-p} \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge^p B, \Delta} \wedge_R^p$$

$$\left| \frac{\Gamma, A \vdash \Delta \quad \oplus^p \quad \Gamma, B \vdash \Delta}{\Gamma, A \wedge^p B \vdash \Delta} \wedge_L^p \quad \left| \frac{\Gamma \vdash A, \Delta \quad \oplus^p \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \vee^p B, \Delta} \vee_R^p \right.$$

$$\frac{\infty}{\perp \vdash} \perp_L \quad \frac{\infty}{\vdash \top} \top_R$$

hyperseq calculi

p QLL: structural schema

In fact, EFQ might be considered a catch-all axiom for external structural rules, as can be evidenced by the proof of the following:

Lemma 3.7 (Admissibility of the structural schema). *Consider the algebraic structure defined in (17), and suppose $t(x_1, \dots, x_n)$ and $s(x_1, \dots, x_n)$ be terms¹² of it. If $|t| \leq |s|$ pointwise when evaluated¹³ in $[0, \infty]_{\oplus p, \otimes}$, then the rule*

$$\frac{t(\Gamma_1 \vdash \Delta_1, \dots, \Gamma_n \vdash \Delta_n)}{s(\Gamma_1 \vdash \Delta_1, \dots, \Gamma_n \vdash \Delta_n)} \quad (\text{str})$$

is admissible in any p -hard counary calculus admitting EFQ .

p QLL: CUT-elimination

Theorem 4.1 (Cut-elimination). *Let $p\text{QLL}^{\text{CUT}}$ be the calculus obtained by removing CUT from $p\text{QLL}$. Then*

$$|\Gamma \vdash \Delta|_{p\text{QLL}} = |\Gamma \vdash \Delta|_{p\text{QLL}^{\text{CUT}}}. \quad (27)$$

In other words, $p\text{QLL}^{\text{CUT}}$ admits CUT .

Proof: *qualitatively* the same as for **MALL**, though we must make sure the rewrite is *validity non-decreasing*, e.g.

$$\begin{array}{c}
 \begin{array}{c}
 \vdots \pi_1 \quad \vdots \pi_2 \\
 \Gamma \vdash A, \Delta \oplus^p \Gamma \vdash B, \Delta \\
 \hline
 \Gamma \vdash A \vee^p B, \Delta
 \end{array}
 \quad \vee_R^p \quad \otimes \quad
 \begin{array}{c}
 \vdots \pi_3 \quad \vdots \pi_4 \\
 \Gamma', A \vdash \Delta' \oplus^{-p} \Gamma', B \vdash \Delta' \\
 \hline
 \Gamma', A \vee^p B \vdash \Delta'
 \end{array}
 \quad \vee_L^p \\
 \hline
 \Gamma, \Gamma' \vdash \Delta, \Delta' \quad \text{CUT}
 \end{array}
 \begin{array}{l}
 \dashrightarrow \\
 \dashrightarrow
 \end{array}
 \begin{array}{c}
 \begin{array}{c}
 \vdots \pi_1 \quad \vdots \pi_3 \\
 \Gamma \vdash A, \Delta \otimes \Gamma', A \vdash \Delta' \\
 \hline
 \Gamma, \Gamma' \vdash \Delta, \Delta'
 \end{array}
 \quad \text{CUT} \\
 \begin{array}{c}
 \vdots \pi_2 \quad \vdots \pi_4 \\
 \Gamma \vdash B, \Delta \otimes \Gamma', B \vdash \Delta' \\
 \hline
 \Gamma, \Gamma' \vdash \Delta, \Delta'
 \end{array}
 \quad \text{CUT}
 \end{array}
 \end{array}$$

And indeed:

$$(|\pi_1| \oplus^p |\pi_2|) \otimes (|\pi_3| \oplus^{-p} |\pi_4|) \leq (|\pi_1| \otimes |\pi_3|) \vee (|\pi_2| \otimes |\pi_4|).$$

p QLL: CUT-elimination

Theorem 4.1 (Cut-elimination). *Let $p\text{QLL}^{\text{CUT}}$ be the calculus obtained by removing CUT from $p\text{QLL}$. Then*

$$|\Gamma \vdash \Delta|_{p\text{QLL}} = |\Gamma \vdash \Delta|_{p\text{QLL}^{\text{CUT}}}. \quad (27)$$

In other words, $p\text{QLL}^{\text{CUT}}$ admits CUT .

subformula property = finitely many proofs to check to determine validity

Theorem 4.2 (Consistency). *For every $0 < p \leq \infty$, $p\text{QLL}$ is consistent, i.e. there are sequents whose provability is 0.*

PROOF. In $p\text{QLL}^{\text{CUT}}$, $\vdash \perp$ can only possibly be proved via EFQ , which has validity 0.

Theorem 4.3 (Decidability). *For any $p\text{QLL}$ sequent $\Gamma \vdash \Delta$, the supremum defining provability is in fact a maximum, i.e. there is a proof π that realizes its provability.*

Hard & qualitative p QLL

In the additive qualitative mode, p QLL is conservative and adequate over **MALL**, and at $p = \infty$, also in the multiplicative mode:

Theorem 4.4. *Let $\Gamma \vdash \Delta$ be a sequent in qualitative p QLL, then*

$$\Gamma \vdash \Delta \text{ provable in } \mathit{hyper}\text{-MALL}^{\mathit{isomix}} \Leftrightarrow |\Gamma \vdash \Delta|_{p\text{QLL}} > 0.$$

Theorem 4.7. *Let $\Gamma \vdash \Delta$ be a sequent in $\text{MALL}^{\mathit{iso}}$ or ∞QLL , then*

$$\Gamma \vdash \Delta \text{ provable in } \text{MALL}^{\mathit{iso}} \Leftrightarrow |\Gamma \vdash \Delta|_{\infty\text{QLL}} \geq 1. \quad (30)$$

Theorem 4.8 (Approximation Theorem). *Let $\Gamma \vdash \Delta$ be a sequent in 1QLL , then*

$$|\Gamma^p \vdash \Delta^p|_{p\text{QLL}} \xrightarrow{p \rightarrow \infty} |\Gamma^\infty \vdash \Delta^\infty|_{\infty\text{QLL}}. \quad (31)$$

η -expansion in $pQLL$

$$\frac{1}{a \wedge^p b \vdash a \wedge^p b} \text{AX} \quad \longrightarrow \quad \frac{\frac{\frac{1}{a \vdash a} \text{AX} \quad \oplus^p \quad b \vdash a}{a \wedge^p b \vdash a} \wedge_L^p \quad \oplus^{-p} \quad \frac{\frac{a \vdash b \quad \oplus^p \quad b \vdash b}{a \wedge^p b \vdash b} \wedge_L^p \quad \text{AX}}{a \wedge^p b \vdash a \wedge^p b} \wedge_R^p}{\vdots? \quad \vdots?}$$

η -expansion in p QLL

$$\frac{1}{a \wedge^p b \vdash a \wedge^p b} \text{AX} \quad \longrightarrow \quad \frac{\frac{1}{a \vdash a} \text{AX} \quad \frac{\vdots?}{b \vdash a} \text{AX}}{a \wedge^p b \vdash a} \wedge_L^p \quad \frac{\frac{\vdots?}{a \vdash b} \text{AX} \quad \frac{1}{b \vdash b} \text{AX}}{a \wedge^p b \vdash b} \wedge_L^p}{a \wedge^p b \vdash a \wedge^p b} \oplus^{-p} \wedge_R^p$$



prelinearity

$$|A \vdash B|^* \leq |B \vdash A|$$

Lemma 4.9. For $0 < p < \infty$,
 $1 \leq (1 \oplus^p x) \oplus^{-p} (y \oplus^p 1) \iff x^* \leq y.$
 with equality when $x^* = y$. For $p = \infty$ the inequality holds regardless
 of the values of x and y . (34)

Algebraic semantics

Softales = ‘enriched’ residuated algebras

Definition 5.2 (Softale). A (p) -softale is a $[0, \infty]_{\otimes}$ -enriched pre-order S equipped with

multiplicative structure: a **unit** object $1 \in S$ and a **tensor** map $\bullet : S \times S \rightarrow S$ satisfying

$$(a \sqsubseteq b) \otimes^* (c \sqsubseteq d) \leq (a \bullet c) \sqsubseteq (b \bullet d), \quad (38)$$

$$a \bullet 1 \cong_{\geq 1} a, \quad (a \bullet b) \bullet c \cong_{\geq 1} a \bullet (b \bullet c), \quad a \bullet b \cong_{\geq 1} b \bullet a, \quad (39)$$

and a **duality** $(-)^{\perp} : S \rightarrow S$ satisfying

$$1^{\perp} = 1, \quad a^{\perp\perp} = a, \quad (40)$$

$$(a \sqsubseteq b) = (b^{\perp} \sqsubseteq a^{\perp}), \quad (41)$$

$$(a \bullet b \sqsubseteq c^{\perp}) = (a \sqsubseteq (b \bullet c)^{\perp}); \quad (42)$$

additive structure: a **bottom** $\perp \in S$, and a choice of **universal p -join** $a \vee^p b$ for all $a, b \in S$:

$$(\perp \sqsubseteq a) = \infty, \quad (43)$$

$$(a \sqsubseteq c) \oplus^{-p} (b \sqsubseteq c) \leq (a \vee^p b \sqsubseteq c), \quad (44)$$

$$(c \sqsubseteq a) \oplus^p (c \sqsubseteq b) \leq (c \sqsubseteq a \vee^p b), \quad (45)$$

Algebraic semantics

Definition 5.9. Let V be a set of propositional variables. The **syntactic p -softale** $\text{LT}^p[V]$ is defined as follows:

- (1) its elements are the formulae of [Definition 3.5](#),
- (2) its order relation is defined as

$$A \Rightarrow B := |A \vdash B|, \quad (52)$$

where the right hand side denotes the **provability** of a sequent;

- (3) its multiplicative and additive structures are given by the connectives of **p QLL**, i.e. respectively by \otimes , $(-)^*$, and \vee^p .

(?) complete for $\llbracket _ \rrbracket_v$?
(in \mathcal{Q})

Proposition 5.10. *Syntactic softales are (pseudo)free, i.e.*

$$\text{Sftl}(\text{LT}^p[V], \mathcal{S}) \cong \text{Set}(V, \mathcal{S}). \quad (54)$$

Therefore for any valuation $v : V \rightarrow S$ of the propositional variables, we get a unique interpretation $\llbracket _ \rrbracket_v$ that exhibits soundness.

Corollary 5.10.1 (Soundness & Completeness). *p QLL is sound and complete for softales, i.e. for every p QLL sequent $\varphi \vdash \psi$, any p -softale and any valuation $v : V \rightarrow S$, we have*

$$|\varphi \vdash \psi| \leq (\llbracket \varphi \rrbracket_v \sqsubseteq \llbracket \psi \rrbracket_v). \quad (55)$$

and in fact the left hand side is a minimum:

$$\bigwedge_{S \in \text{Sftl}} \bigwedge_{v: V \rightarrow S} (\llbracket \varphi \rrbracket_v \sqsubseteq \llbracket \psi \rrbracket_v) = |\varphi \vdash \psi|. \quad (56)$$

Application: Bayesian probability

Now consider a valuation $v : V \rightarrow [0, \infty]$. The associated **prelinear theory** \mathbb{T}_v has constants the elements of V and axioms

$$\left| \frac{p(a)}{\vdash a} a_R \quad \frac{p(a)^*}{a \vdash} a_L \quad \text{for all } a \in V. \quad (59) \right.$$

Starting from a finite set Ω of *outcomes*, endowed with a probability density $p : \Omega \rightarrow [0, 1]$, consider the prelinear theory (59) with atoms Ω evaluated by p . An event $A \subseteq \Omega$ corresponds to a formula in this theory, namely $\sum_{a \in A} a$, which we denote²³ again by A .

Every sequent $A \vdash B$ where A, B are such propositions correspond then to the relative probability of B compared to A , i.e. the *odds* of B to A . In particular the judgment $A \vdash A \cap B$ is so proven

$$\frac{\frac{\frac{\bigoplus_{b \in A \cap B} \bigoplus_{a \in A}^* p(a)^* \otimes^* p(b)}{\bigoplus_{b \in A \cap B} \bigoplus_{a \in A}^* (a \vdash \otimes^* \vdash b)} \bigoplus_b \bigoplus_a^* (a_L \otimes^* b_R)}{\bigoplus_b \bigoplus_a^* \text{MIX}^*} \bigoplus_b \bigoplus_a^* \text{MIX}^*}{\frac{\bigoplus_{b \in A \cap B} \bigoplus_{a \in A}^* a \vdash b}{\bigoplus_{b \in A \cap B} A \vdash b} \bigoplus_b \vee_L^1} \bigoplus_b \vee_R^1} \quad (60)$$

where the top reduces to

$$\frac{\bigoplus_{b \in A \cap B} p(b)}{\bigoplus_{a \in A} p(a)} = \frac{p(A \cap B)}{p(A)} = p(B | A). \quad (61)$$

$$|a \vdash b| = \text{odds of } a : b$$

$$A : A \cap B = B | A$$

