The Reasonable Effectiveness of Linear Logic in Quantitative Methods

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Quantitative Linear Logic

Linear Quantitative Logic

What is Quantitative Logic?

According to **Lawvere**'s *"Metric Spaces, Generalized Logic, and Closed Categories"*:

> *"Logic signifies formal relationships which are general in character, we may more precisely identify logic with this scheme of interlocking adjoints and then observe that all of logic applies directly to structures valued in arbitrary closed categories $\mathcal{V}$, for example in quantitative logic $\mathcal{V} = \mathbb{R}$."*

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This distinguishes it from *logics* **about** *quantity*, i.e. **fuzzy logics** (Zadeh, Yager, Goguen etc.), **continuous model theory** (Yaacov, Berenstein, etc.), or the **logics of the Lawvere quantale** (Bacci, Mardare, Panangaden, Plotkin), and more.

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change **enrichment**,

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Bayesian probability, information theory,
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Claim: linear logic is the correct 'blueprint' for such a programme!

The multiplicative reals

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$[0,\infty]_{\oplus,\otimes}$ with the following algebraic structure on $([0,\infty],\leq)$:

	polarity	**additive**	**multiplicative**
duality	**positive**		
	negative		

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		negative	$\top := \infty$ $a \oplus^* b$	$\mathbf{1} := 1$ $a \otimes^* b$	$\infty \otimes^* 0 := \infty$

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$\otimes$ and $\otimes^*$ are multiplication (conjunctive and disjunctive),

$\oplus$ is addition,

$\oplus^*$ is harmonic sum: $a \oplus^* b = (a^* \oplus b^*)^*$,

The multiplicative p-reals

$[0,\infty]_{\oplus^p,\otimes}$ with the following algebraic structure on $([0,\infty],\leq)$:

distributes over

		polarity	**additive**		**multiplicative**	
duality $a^* := 1/a$	**de Morgan**	**positive**	$\bot := 0$, $a \oplus^p b$	$\xrightarrow{p\to+\infty}$ $\bot := 0$, $a \vee b$	$\mathbf{1} := 1$, $a \otimes b$	$\infty \otimes 0 := 0$
		negative	$\top := \infty$, $a \oplus^{-p} b$	$\xrightarrow{p\to+\infty}$ $\top := \infty$, $a \wedge b$	$\mathbf{1} := 1$, $a \otimes^* b$	$\infty \otimes^* 0 := \infty$

$\underbrace{a \oplus^{-p} b}_{\text{soft}}$ $\qquad$ $\underbrace{a \wedge b}_{\text{hard}}$

where

$\otimes$ and $\otimes^*$ are multiplication (conjunctive and disjunctive),

$\oplus^p$ is p-sum: $a \oplus^p b = (a^p \oplus b^p)^{1/p}$, $0 < p \leq \infty$,

$\oplus^{-p}$ is harmonic p-sum: $a \oplus^{-p} b = (a^* \oplus^p b^*)^*$, QLL

Quantitative sequent calculi

A p-**quantitative sequent calculus** is a sequent calculus with $\mathcal{V} = [0, \infty]_{\oplus^p, \otimes}$, thus **all notions of (proof, sequent, semantic) validity live in** $[0, \infty]_{\oplus^p, \otimes}$.

Quantitative sequent calculi

Rules must explicitly say how to combine the premises to get the conclusion:

e.g.

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

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$$[0, \infty] \ni \left| \begin{array}{c} \vdots\, \pi \\ \Gamma \vdash \Delta \end{array} \right| = \begin{cases} r & \text{if} \quad \pi = \dfrac{r}{\Gamma \vdash \Delta} \\ & \end{cases}$$

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Quantitative sequent calculi

The **provability** of a sequent is thus defined in terms of validities of its proofs:

$$|\Gamma \vdash \Delta| = \bigvee \left\{ |\pi| \;\middle|\; \begin{array}{c} \vdots\, \pi \\ \Gamma \vdash \Delta \end{array} \right\}$$

Note that when $\mathcal{V} = \mathbf{Prop}$, this recovers the classical definition.

Structural Rules

$$\frac{\color{red}{1}}{A \vdash A}\ \mathsf{AX} \qquad \frac{\Gamma \vdash A \quad \color{red}{\otimes} \quad A \vdash \Delta}{\Gamma \vdash \Delta}\ \mathsf{CUT}$$

$$\frac{\color{red}{1}}{\vdash}\ \mathsf{EMP} \qquad \frac{\Gamma \vdash \Delta \quad \color{red}{\otimes} \quad \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \mathsf{MIX}$$

$$\frac{\color{red}{0}}{\Gamma \vdash \Delta}\ \mathsf{EFQ}$$

Things to observe:

1. EFQ

pQLL

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$$\frac{\Gamma \vdash A, \Delta}{\Gamma, A^{\perp} \vdash \Delta}\ (-)^{\perp}_{\mathsf{L}} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^{\perp}, \Delta}\ (-)^{\perp}_{\mathsf{R}}$$

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Additive Connectives

$$\frac{\infty}{\Gamma, \bot \vdash \Delta}\ \bot_{\mathsf{L}}$$

$$\frac{A \vdash \Delta \quad \oplus^{p} \quad B \vdash \Delta}{A \wedge B \vdash \Delta}\ \wedge_{\mathsf{L}}$$

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Open Question: *exponentials?*

One proof

$$
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$$

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Theorem

$\Gamma \vdash \Delta$ *is provable in 'isomix' MALL if and only if* $|\Gamma \vdash \Delta|_{pQLL} > 0$.

Proof-theoretic results

Theorem (CUT-elimination, in (Capucci, Atkey, Grellois, Komendantskaya, and Daggitt 2026))

CUT is eliminable, that is $|\Gamma \vdash \Delta|_{pQLL} = |\Gamma \vdash \Delta|_{pQLL^{\cancel{CUT}}}$.

Proof.

The usual, paired with arguments that show that each proof rewrite is *validity non-decreasing*.

In the proof, we use that $\otimes$ distributes over $\oplus^{p}/\oplus^{-p}$, unitality and associativity of $\otimes$, etc.

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Open Question: *what is a quantitative proof net?*

Model theory

pQLL is **sound and complete** for a class of $[0,\infty]_\otimes$-enriched preorders called **softales**—roughly, $*$**-autonomous 'soft lattices'** ($[0,\infty]_{\oplus^p,\otimes}$ is the ur-example).

$$|A \vdash B| = \bigwedge_{[\![-]\!]:V\to\mathcal{S}} [\![A]\!] \sqsubseteq_{\mathcal{S}} [\![B]\!].$$

This idea goes back to (Lawvere 1973): **categorical semantics of quantitative logic is quantitative (= $[0,\infty]_\otimes$-enriched) categories**.

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Open Question: *is there an analogoue of phase/coherence space semantics?*

Open Question: *can we tighten completeness to a smaller class of models? What does it take to target $[0,\infty]_{\oplus,\otimes}$?*

Application: Bayesian probability

Every atomic probability space $p : \Omega \to [0,1]$ yields a theory with propositional constants $\ulcorner \omega \urcorner \in \Omega$ and axioms

$$\frac{p(a)}{\vdash a}\ a_{\mathsf{R}} \qquad \frac{p(a)^*}{a \vdash}\ a_{\mathsf{L}} \qquad \text{for all } a \in V.$$

Every event $A \subseteq \Omega$ corresponds to a formula in this theory, which we abbreviate as $\bigvee_{a \in A} a$, and denote again by A.

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Every event $A \subseteq \Omega$ corresponds to a formula in this theory, which we abbreviate as $\bigvee_{a \in A} a$, and denote again by A. Then one can compute $p(B \mid A)$ by computing the odds $A \vdash A \cap B$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\bigoplus_{b \in A \cap B} \bigoplus^*_{a \in A}\ p(a)^* \otimes p(b)}{\bigoplus_{b \in A \cap B} \bigoplus^*_{a \in A}\ (a \vdash \otimes \vdash b)}\ \bigoplus_b \bigoplus^*_a (a_{\mathsf{L}} \otimes b_{\mathsf{R}})}{\bigoplus_{b \in A \cap B} \bigoplus^*_{a \in A}\ a \vdash b}\ \bigoplus_b \bigoplus^*_a \mathsf{MIX}}{\bigoplus_{b \in A \cap B}\ A \vdash b}\ \bigoplus_b \vee_{\mathsf{L}}}{A \vdash A \cap B}\ \vee_{\mathsf{R}}$$

and we indeed have $\bigoplus_{b \in A \cap B} \bigoplus^*_{a \in A}\ p(a)^* \otimes p(b) = \dfrac{p(A \cap B)}{p(A)} = p(B \mid A)$.

Applications: property-driven verification

Property-driven training

Goal: Have a neural model $f : \Theta \times X \longrightarrow Y$ satisfy a spec φ.

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adequacy: How do we formally link the minimisation of $[\![\varphi]\!]$ with the satisfaction of φ?

Logspace & gradients

Call **logspace** the poset $([-\infty, +\infty], \geq)$ equipped with addition and (harmonic) **log-sum-exp**:

$$a \cup^1 b = -\log(e^{-a} + e^{-b}).$$

It also has a duality $-a$ which makes it into a softale, with enriched relation $a \sqsubseteq b := e^{a-b}$.

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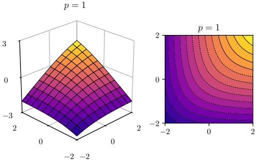

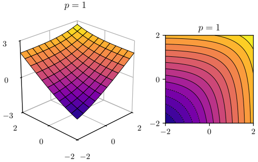

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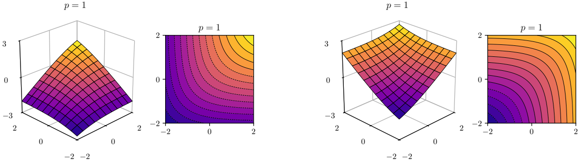

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It is well-known that $([-\infty, +\infty], 0, +)$ is a Lie group whose Lie algebra is $([0, \infty], 1, \otimes)$.

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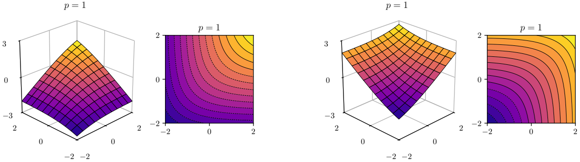
This is the **natural model** for property-driven training: losses live in logspace!

Log-sum-exp and its de Morgan dual have excellent gradients:



$$\frac{\partial(x \cup^1 y)}{\partial x} = \frac{e^y}{e^x + e^y} = e^{-y} \multimap (e^{-x} \oplus^* e^{-y}),$$

$$\frac{\partial(x \cap^1 y)}{\partial x} = \frac{e^x}{e^x + e^y} = e^{-x} \multimap (e^{-x} \oplus^* e^{-y})$$

It is well-known that $([-\infty, +\infty], 0, +)$ is a Lie group whose Lie algebra is $([0, \infty], 1, \otimes)$.

Open Question: *does this have anything to do with differential LL?*

Challenges:

effectiveness: How do we construct $[\![-]\!]$ as to balance logical and analytical properties?

adequacy: How do we formally link the minimisation of $[\![\varphi]\!]$ with the satisfaction of φ?

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(as demonstrated in Flinkow, Komendantskaya, Capucci, and Monahan 2026)

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adequacy: How do we formally link the minimisation of $[\![\varphi]\!]$ with the satisfaction of φ?

$\rightsquigarrow$ **quantitativity + 'completeness'** means we can tie $[\![\varphi]\!]$ directly to its validity in the logspace model and provability in a theory

Acknowledgments

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Bob Atkey

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Charles Grellois

Matthew Daggitt

Adequate Losses via Quantitative Linear Logic

(http://arxiv.org/abs/2605.13348)

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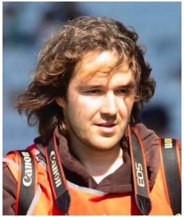

Thomas Flinkow

Katya Komendantskaya

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Rosemary Monahan

Quantitative Linear Logic for Neuro-Symbolic Learning and Verification

(https://arxiv.org/abs/2605.13845)

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Next week at FLoC!

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Alessio Coltellacci

Alessandro Bruni

Katya Komendantskaya

Kathrin Stark

Towards Quantitative Logics in Rocq

(`https://coq-workshop.gitlab.io/2026/files/EA1.pdf`)

Thanks!

References I

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